

Lemma: Let  $1 \leq p \leq \infty$  &  $\frac{1}{p} + \frac{1}{q} = 1$ . Suppose  $g$  is a measurable function on a  $\sigma$ -finite measure space  $(X, \mathcal{S}, \mu)$  such that

$$(*) M_g = \sup \left\{ \int_X f \cdot g : f \text{ simple} \wedge \|f\|_p = 1 \right\} < \infty.$$

Then  $g \in L^q$  and  $M_g = \|g\|_q$ .

Proof: Since  $g$  is measurable, there is a sequence of simple functions  $g_n \rightarrow g$  p.w. &  $\|g_n\|_1 \leq \|g\|_1$ . Given that  $X$  is  $\sigma$ -finite, there are countably many  $E_n$  of finite measure s.t.  $E_n \uparrow X$ . Hence,  $g_n = g_n \chi_{E_n} \rightarrow g$  &  $\|g_n\|_1 \leq \|g\|_1$ .

Since  $\left| \int_X f g \right| \leq M_g$ , &  $f$  simple &  $\|f\|_p = 1$ , by replacing  $f$  with  $f g_n / \|g_n\|_1$  we get

$$\int_X |fg| \leq M_g, \text{ whenever } f \text{ simple, } \|f\|_p = 1.$$

$$\text{Then } \int_X |f g_n| \leq \int_X |fg| \leq M_g \quad \dots \quad (1)$$

Case I  $q < \infty$ : For  $f = \chi_{E_n} / \mu(E_n)^{1/p}$ ,  $\|f\|_p = 1$ .

Hence,  $\int_{E_n} |g_n| \leq M_g \mu(E_n)^{1/p} < \infty$ . That is,

$g_n \in L^q(X, \mathcal{S}, \mu)$  and  $g_n$  is simple, thus,  $g_n \in L^q(X, \mathcal{S}, \mu)$ , (because every simple  $L^p$  function is in any  $L^q$ ).

$$(\because \|f_n\|^2 = \int \sum_{i=1}^n |k_i|^2 \chi_{E_i} = \sum |k_i|^2 \mu(E_i) < \infty).$$

Further, consider  $f_n = \frac{g_n}{\|g_n\|_2^{2-1}}$ , then  $\|f_n\|_p = 1$  and  $f_n$  is simple. Hence, from (1), we get

$$\|g_n\|_2 \leq M_g.$$

By Fatou's lemma,

$$\int g^2 \leq \liminf \int |g_n|^2 \leq M_g^2 \Rightarrow \|g\|_2 \leq M_g < \infty.$$

that  $\|g\|_2 = \sqrt{\int g^2}$ . By Holder's inequality

$$|\int_X f_2| \leq \int_X |fg| \leq \|f\|_p \|g\|_2 = \|g\|_2. \text{ if } \|f\|_p = 1.$$

$$\Rightarrow M_g = \sup |\int_X fg| \leq \|g\|_2.$$

Case 3 =  $\infty$ : In this case  $p=1$ , and we can

$$\text{take } f_n = \frac{g_n}{\|g_n\|_\infty}, \quad \|f_n\|_1 =$$

Consider the set  $E = \{x \in X : |g(x)| > M_g + \epsilon\}$

having the measure, for each  $\epsilon > 0$ .

Since  $(X, \mathcal{S}, \mu)$  is  $\sigma$ -finite, w.l.g. we can assume that  $0 < \mu(E) < \infty$ . Setting

$$f = \frac{\chi_E g_n}{\mu(E)}. \text{ Then } \|f\|_1 = 1, \text{ and}$$

$$M_g \geq |\int_X fg| = \int_X fg = \int_E \frac{\chi_E g_n}{\mu(E)} \geq M_g + \epsilon, \text{ is}$$

impossible. Hence  $|g(x)| \leq M_g$  a.e.

$\Rightarrow \|g\|_\infty \leq M_g$ . Again by Holder's  
 inequality,  $|Sf(y)| \leq \|f\|_1, \|g\|_\infty = \|g\|_1$ , if  $\|f\|_1 = 1$ .  
 $\Rightarrow M_g \leq \|g\|_1$ .

Remark: From the proof, it is clear that, to prove  
 $g \in L^2$ , (or  $M_g = \|g\|_2$ ), it is enough to  
 take supremum in (\*) on those  $L^1$ -simple  
 functions which vanish over the set of  
 finite measure.

## Product measure:

Let  $(X, \mathcal{S}, \mu)$  and  $(Y, \mathcal{T}, \nu)$  be two measurable spaces. For  $A \in \mathcal{S}$ , and  $B \in \mathcal{T}$ , we say  $A \times B$  is a measurable rectangle.

Let  $\mathcal{Q}$  be the collection of all finite unions of rectangles from  $\mathcal{S} \times \mathcal{T}$ . Then  $\mathcal{Q}$  is an algebra of sets from  $\mathcal{S} \times \mathcal{T}$ . Now, for

$$A \times B = \bigcup_{i,j=1}^{\infty} A_i \times B_j, \text{ we can write}$$

$$\chi_A(x) \chi_B(y) = \chi_{A \times B}(x, y) = \sum_{i,j=1}^{\infty} \chi_{A_i \times B_j}(x, y).$$

Integrating both the sides, first w.r.t.  $X$  and then w.r.t.  $Y$  and using Lebesgue-Levi theorem, we get

$$\mu(A) \nu(B) = \sum_{i,j=1}^{\infty} \mu(A_i) \nu(B_j). \quad (*)$$

Now, for  $E \in \mathcal{Q}$  with  $E = \bigcup_{i,j=1}^{m,m} A_i \times B_j$ , define

$$\gamma(E) := \sum_{j=1}^m \sum_{i=1}^m \mu(A_i) \nu(B_j).$$

Then by (\*),  $\gamma$  is a pre-measure on  $\mathcal{Q}$ .

Hence  $\gamma$  generates an outer measure  $\gamma^*$  on  $P(X \times Y)$ . Let

$$\begin{aligned} S \otimes T &= \{ \gamma^* \text{-measurable sets in } X \times Y \} \\ &= \text{the smallest } \sigma\text{-algebra containing } \mathcal{Q}. \end{aligned}$$

Then  $\gamma = \eta^*/SOT : SOT \rightarrow [0, \infty]$  is  
a measure on  $SOT$ .

Notice that if  $(X, \mathcal{S}, \mu)$  &  $(Y, \mathcal{T}, \nu)$  are  $\sigma$ -finite, then  $X = \cup E_m$ ,  $Y = \cup F_m$ , where  $\mu(E_m) < \infty$  &  $\nu(F_m) < \infty$ . Hence

$$\begin{aligned}\gamma(X \times Y) &= \sum \sum \gamma(E_m \times F_m) = \sum \sum \gamma_0(E_m \times F_m) \\ &= \sum \sum \mu(E_m) \nu(F_m).\end{aligned}$$

Hence,  $\gamma$  will be a unique extension of  $\gamma_0$  such that  $\gamma(A \times B) = \mu(A) \nu(B)$ , whenever  $A \times B \in \mathcal{Q}$ . (See the previous result page 73.)

In this case, we write  $\gamma = \mu \times \nu$ . Hence  $(X \times Y, SOT, \mu \times \nu)$  is a measure space.

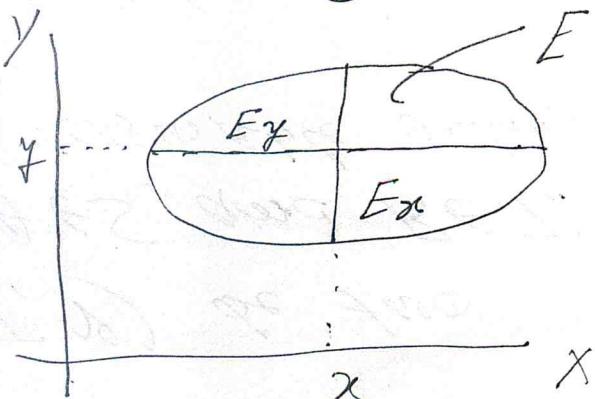
Note that if  $E \in SOT$ , then  $E$  need not be a rectangle, but we need to calculate  $(\mu \times \nu)(E)$ !

For this, we define projection (or Section) of  $E$  on  $X$  &  $Y$ , in the following way.

For  $(x, y) \in X \times Y$ , let

$$E_x = \{y \in Y : (x, y) \in E\}$$

$$\text{and } E_y = \{x \in X : (x, y) \in E\}.$$



Monotone class: A collection of sets in  $X$  which is closed under countable increasing union and countable decreasing intersection is called monotone class. That is  $M$  is a monotone class if  $A_i \uparrow$  &  $B_j \downarrow$  seq's in  $M$ , implies  $\cup A_i$  &  $\cap B_j \in M$ .

Ex. Every  $\sigma$ -algebra is a monotone class, but converse need not be true. However, if  $\mathcal{A}$  is a  $\sigma$ -algebra of sets in  $X$ . Then the monotone class generated by  $\mathcal{A}$  coincides with the  $\sigma$ -algebra generated by  $\mathcal{A}$ .

Let  $S(\mathcal{A}) = \sigma$ -algebra generated by  $\mathcal{A}$  and  $M(\mathcal{A}) = \text{monotone class generated by } \mathcal{A}$

Theorem (Monotone class theorem):

$$S(\mathcal{A}) = M(\mathcal{A}).$$

Proof: Since  $S(\mathcal{A})$  is a monotone class containing  $\mathcal{A}$ ,  $M(\mathcal{A}) \subset S(\mathcal{A})$ . On the other hand we need to show that if  $M(\mathcal{A})$  is a  $\sigma$ -algebra, then  $M(\mathcal{A}) \supset S(\mathcal{A})$ . For this, let  $E \in M(\mathcal{A}) = M$ . By) and define  $M(E) = \{F \in M : E \cap F, F \setminus E, E \setminus F \text{ are in } M\}$

(i)  $\emptyset, E \in \mathcal{P}(E)$  ( $\because E \in M$ ).

(ii)  $\mathcal{P}(E)$  is a monotone class,

if  $F_n \in M(E) \forall n$ , then  $F_n \setminus E \in M \forall n$

$$\Rightarrow \bigcup_{n=1}^{\infty} (F_n \setminus E) = \bigcup_{n=1}^{\infty} (F_n \cap E^c) \in M(E) \text{ etc.}$$

(iii) If  $E \in A$ , then  $F \in M(E)$ ,  $\forall F \in A$ ,  
because  $A$  is an algebra.

i.e.  $E \in A \Rightarrow A \subseteq M(E)$ . Again, since  
 $M(E)$  is a monotone class,

$$M = M(A) \subseteq M(E), \forall E \in A.$$

Notice that  $E \in M(F) \iff F \in M(E)$  (by def).

Hence for  $F \in M$ ,  $F \in M(E)$ ,  $\forall E \in A$

$$\Rightarrow E \in M(F), \forall E \in A$$

$$\Rightarrow A \subseteq M(F), \forall F \in M$$

$$\Rightarrow M \subseteq M(F), \forall F \in M.$$

thus, if  $E, F \in M$ , then  $E \setminus F, E \cap F \in M$ .

Since  $A \subseteq M$  &  $X \in A$ , it follows that  
 $M$  is closed under union and complement.  
that is,  $M$  is an algebra.

now, if  $\{A_i\}_{i=1}^{\infty} \subseteq M$ , then  $\bigcup_{i=1}^{\infty} A_i \in M \forall i$ ,

but  $M$  is a monotone class, implies

$\bigcup_{i=1}^{\infty} A_i \in M$ . That is,  $M$  is a  $\sigma$ -algebra  
and  $M(A) = S(A)$ .

Theorem: Let  $(X, \mathcal{S}, \mu)$  and  $(Y, \mathcal{T}, \nu)$  be two  $\sigma$ -finite measure spaces. Suppose  $E \in \mathcal{S} \otimes \mathcal{T}$ , then (i)  $E_x \in \mathcal{S}$ , and  $E^y \in \mathcal{T}$ ,  $\forall (x, y) \in X \times Y$ .  
(ii)  $x \mapsto \nu(E_x)$  and  $y \mapsto \mu(E^y)$  are measurable functions on  $(X, \mathcal{S})$  and  $(Y, \mathcal{T})$  respectively.  
(iii)  $(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$ .

Proof: The proof of this result is based on monotone class theorem.

(i) Let  $\mathcal{Q} = \{E \in \mathcal{S} \otimes \mathcal{T} : E_x \in \mathcal{S}, E^y \in \mathcal{T}, \forall (x, y) \in X \times Y\}$ . Then  $\mathcal{Q} \subseteq \mathcal{S} \otimes \mathcal{T}$ . On the other hand, it is easy to see that  $\mathcal{Q}$  is a  $\sigma$ -algebra containing  $\mathcal{R}$ . Hence  $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{Q}$ . This proves (i).

Proof of (ii) & (iii): Let  $\mu$  &  $\nu$  be both finite measures. Let  $\mathcal{P} = \{E \in \mathcal{S} \otimes \mathcal{T} : (ii) \& (iii) \text{ hold}\}$ .  
Let  $E = A \times B$ ,  $A \in \mathcal{S}$ ,  $B \in \mathcal{T}$ .  
For  $E = A \times B$ ,  $\nu(E_x) = \nu(B)$  and  $\mu(E^y) = \mu(A)$ .  
 $\nu(E_x) = \chi_A(x) \nu(B)$  and  $\mu(E^y) = \mu(A) \chi_B$ .  
and  $(\mu \times \nu)(E) = \mu(A) \nu(B) = \int \nu(E_x) d\mu(x)$   
 $= \int \mu(E^y) d\nu(y)$ .

Thus, Theorem is true for  $E = A \times B$ . Hence, it will hold for finite disjoint union of

rectangles. Hence by the monotone class theorem, it is suffice to show that  $\mathcal{P}$  is a monotone class.

If  $E_n$  is an  $\cap$  seg'n in  $\mathcal{P}$ , then  $E = \bigcup_{n=1}^{\infty} E_n$ , then  $\mu((E_n)^y)$  is measurable function that increases p.w. to  $\mu(E^y)$ . Hence  $\mu(E^y)$  is countable and by MCT,

$$(*) \quad \int \mu(E^y) d\nu(y) = \lim \int \mu((E_n)^y) d\nu(y) \\ = \lim (\mu \times \nu)(E_n) = (\mu \times \nu)(E).$$

Akewise,  $\mu \times \nu(E) = \int \nu(E_x) d\mu(x)$ , so  $E \in \mathcal{P}$ .

Similarly, if  $E_n \downarrow$  seg'n in  $\mathcal{P}$  and  $E = \bigcap_{n=1}^{\infty} E_n$ , then  $y \mapsto \mu((E_n)^y) \in L^1(\nu)$ , because,  $\mu((E_n)^y) < \mu(X) < \infty$  and  $\nu(Y) < \infty$ .

Since  $\mu((E_n)^y) \downarrow \mu(E^y)$ . By DCT, we

can see that  $E \in \mathcal{P}$ .

Finally, if  $\mu$  and  $\nu$  are  $\sigma$ -finite, then  $X \times Y$  can be consider as the union of an increasing seg'n  $\{X_i \times Y_j\}$  of rectangles of finite measure. By previous argument applied on  $E \cap (X_i \times Y_j)$ , we get

$$\mu \times \nu(E \cap (X_i \times Y_j)) = \int X_i(x) \nu(E_x \cap Y_j) d\mu(x)$$

$$= \int X_{Y_i}(y) \mu(E^y \cap X_i) d\nu(y).$$

By MCT, we get the required result.

Ex. Let  $f: X \xrightarrow{\text{measurable}} R$ . Then we can define

$$\varphi: X \times R \rightarrow R^2 \xrightarrow{\psi} R \quad \text{by}$$

$$\varphi(x, y) = (f(x), y) \text{ and } \psi(\xi, \eta) = \xi - \eta.$$

$$\begin{aligned} \varphi^{-1}\{(a, b) \times (c, d)\} &= \{(x, y) \in X \times R : \varphi(x, y) \in (a, b) \times (c, d)\} \\ &= \{(x, y) : a < f(x) < b, c < y < d\} \\ &= f^{-1}\{(a, b)\} \times (c, d) \text{ is a measurable} \\ &\quad \text{subset of } X \times R. \end{aligned}$$

Hence  $\varphi \circ \varphi$  is measurable. Consider

$$\begin{aligned} (\varphi \circ \varphi)^{-1}(0) &= \{(x, y) \in X \times R : (\varphi \circ \varphi)(x, y) = 0\} \\ &= \{(x, y) : \varphi(f(x), y) = 0\} \\ &= \{(x, y) : y = f(x), x \in X\} \\ &= G_f, \text{ the graph of } f. \end{aligned}$$

Hence, the graph of a measurable function  
is measurable.

Theorem (Tonelli): Let  $(X, \mathcal{S}, \mu)$  and  $(Y, \mathcal{T}, \nu)$  are  $\sigma$ -finit. Let  $f: X \times Y \rightarrow [0, \infty]$  be a SOT-misble function. Then for fixed  $(x_0, y_0) \in X \times Y$ ,

- $x \mapsto f(x, y_0)$  and  $y \mapsto f(x_0, y)$  are measurable functions on  $(X, \mathcal{S})$  and  $(Y, \mathcal{T})$  respectively.

(ii)  $y \mapsto \int f(x, y) d\mu(x)$  and  $x \mapsto \int_y f(x, y) d\nu(y)$  are misble.

$$(iii) \int_{X \times Y} f(x, y) d(\mu \times \nu)(x, y) = \iint_X f(x, y) d\mu(x) d\nu(y)$$

$$= \int_X \int_Y f(x, y) d\nu(y) d\mu(x).$$

Proof: Since  $f$  is SOT-misble on  $X \times Y$ ,  
 $\exists$  a seq<sup>n</sup>  $\varphi_n$  of simple function that increases to  $f$  point-wise. Hence,

$$\lim \varphi_n(x_0, y) = f(x_0, y) \quad \leftarrow \{y\} \text{ are misble.}$$

$$\& \lim \varphi_n(x, y_0) = f(x, y_0) \quad \leftarrow \{x\}$$

Now, by MCT,

$$(1) - y \mapsto \int_X f(x, y) d\mu(x) = \lim_{n \rightarrow \infty} \int_X \varphi_n(x, y) d\mu(x)$$

$$= \lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \alpha_j \nu \{E_j\}_{y \in Y},$$

where  $\varphi_n = \sum_{j=1}^{k_n} \alpha_j X_{E_j}$ . This proves (ii).

Further,  $(\varphi_n)_{y \in Y} = \sum_{j=1}^{k_n} \alpha_j \nu \{E_j\}_{y \in Y} \uparrow$  sequence,

By applying MCT in (1), we get

$$\begin{aligned} \iint_{X \times Y} f(x, y) d\mu(x) d\nu(y) &= \lim_{n \rightarrow \infty} \iint_{X \times Y} g_n(x, y) d\mu(x) d\nu(y). \\ &= \lim_{n \rightarrow \infty} \int_{X \times Y} g_n(x, y) d(\mu \times \nu)(x, y). \end{aligned}$$

Similarly, other equality follows.

Fubini's Theorem:

Let  $(X, \mathcal{S}, \mu)$  and  $(Y, \mathcal{T}, \nu)$  be two  $\sigma$ -finite measure spaces. If  $f \in L^1(\mu \times \nu)$ , then (i)  $x \mapsto f(x, y)$  &  $y \mapsto f(x, y)$  are a.e. integrable on  $X$  &  $Y$  respectively.

(ii)  $y \mapsto \int_X f(x, y) d\mu(x)$  and  $x \mapsto \int_Y f(x, y) d\nu(y)$  are integrable on  $Y$  and  $X$ , respectively.

$$\begin{aligned} \text{(iii)} \quad \iint_{X \times Y} f(x, y) d(\mu \times \nu)(x, y) &= \iint_{X \times Y} f(x, y) d\mu(x) d\nu(y) \\ &= \int_X \int_Y f(x, y) d\nu(y) d\mu(x). \end{aligned}$$

Proof: If  $f = f^+ - f^-$ , then  $f^+, f^- \in L^1(\mu \times \nu)$  and are non-negative. Hence, by linearity of integral on  $L^1$ ,

$$\iint_{X \times Y} f d(\mu \times \nu) = \iint_{X \times Y} f^+ d(\mu \times \nu) - \iint_{X \times Y} f^- d(\mu \times \nu).$$

Hence by Tonelli's theorem,

$$(*) \int \int \int f^*(x,y) d\nu(y) d\mu(x) = \int \int \int f^* d\mu(x) d\nu(y) < \infty.$$

$\Rightarrow \int \int \int f^*(x,y) d\nu(y)$  &  $\int \int f^*(x,y) d\mu(x)$  are finite a.e. w.r.t  $\nu$  and  $\mu$  resp. and integrable w.r.t  $\nu$  and  $\mu$  resp. This proves (i) and (ii). Hence by Tonelli theorem and (i), we get (iii).

Remark: If the measure space  $(X, \mathcal{B}, \mu)$  and  $(Y, \mathcal{T}, \nu)$  are complete, their product  $(X \times Y, \mathcal{S} \otimes \mathcal{T}, \mu \times \nu)$  need not be complete. Suppose  $A \subset X$ ,  $A \neq \emptyset$ ,  $\mu(A) = 0$ . Let  $B \subset Y$  but  $B \notin \mathcal{T}$ , then  $A \times B \subset A \times Y$ , but  $\eta^*(A \times B) \leq \eta^*(A \times Y) = \mu(A)\nu(Y) = 0$ , however,  $A \times B \notin \mathcal{S} \otimes \mathcal{T}$ .

Ex. Let  $m_1^*$  and  $m_2^*$  denote the usual Lebesgue outer measure on  $\mathbb{R}$  and  $\mathbb{R}^2$  resp. and  $m_1, m_2$  are their corresponding Lebesgue measures. Then  $m_1 \times m_1^*$  is not a complete measure, however,  $m_2$  is complete. Though, completion of  $m_1 \times m_1^*$  is  $m_2$ .

Let  $R_2 = \{ (a,b) \times (c,d) : a, b, c, d \in \mathbb{R} \}$ , and

$\mathcal{B}_2$  is the  $\sigma$ -algebra generated by  $R_2$ . Then  $\mathcal{B}_2$  is nothing but Borel  $\sigma$ -algebra on  $\mathbb{R}^2$ .

Since  $R_2 \subset M_1 \otimes M_1$ , it follows that

$$\mathcal{B}_2 \subset M_1 \otimes M_2 \quad (\because M_1 \otimes M_2 \text{ is a } \sigma\text{-ab.})$$

Further,  $R_2 \subset M_2$  and  $M_2$  is the smallest  $\sigma$ -algebra containing  $R_2$ , hence

$$\mathcal{B}_2 \subset M_1 \otimes M_1 \subset M_2.$$

But completion of  $\mathcal{B}_2$  is  $M_2$ . So, if  $E \in M_2$ , then  $\exists F, G \in \mathcal{B}_2$  with  $F \subset E \subset G$  such that  $m_2(G \setminus F) = 0$ . Thus,

$$m_1 \times m_1(E \setminus F) \leq m_1 \times m_1(G \setminus F) = m_2(G \setminus F) = 0.$$

$$(m_1 \times m_1)(E) = (m_1 \times m_1)(F) = m_2(F) = m_2(G).$$

$$\text{Since } m_2(F) \leq m_2(E) \leq m_2(G).$$

$$\Rightarrow (m_1 \times m_1)(E) = m_2(E).$$