DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA731: Linear Algebra and Functional Analysis — End Semester Exam
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Time: 3 hours — Maximum Marks: 50

Linear Algebra

1. For $A \in M_n(\mathbb{C})$, define $\|A\| = \max\left\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A\right\}$. Show that $\|\cdot\|$ is a matrix norm on $M_n(\mathbb{C})$. **[3]**

2. Let $A = [a_{ij}] \in M_n(\mathbb{C})$ be a positive definite matrix such that $a_{ii} = 1$ for all $i = 1, \ldots, n$. Show that $|a_{ij}| \leq 1$ for all $i, j = 1, \ldots, n$. Can equality occur for all $i, j = 1, \ldots, n$? **[2+1]**

3. Let $A \in M_n(\mathbb{C})$. Show that A^*A is unitarily similar to AA^*. **[2]**

4. Let $A = [a_{ij}] \in M_n(\mathbb{C})$ be a positive semi-definite matrix. Show that the matrix $[|a_{ij}|^2]$ is also positive semi-definite. **[2]**

Functional Analysis

5. Let $R : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the right shift operator defined by $R(x) = (0, x_1, x_2, x_3, \ldots)$, for each $x = (x_1, x_2, x_3, \ldots) \in l^2(\mathbb{N})$. Prove that

   (a) resolvent set $\rho(R) = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$. **[1]**

   (b) point spectrum set $\sigma_p(R) = \emptyset$. **[1]**

   (c) continuous spectrum set $\sigma_c(R) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. **[2]**

   (d) residual spectrum set $\sigma_r(R) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. **[2]**

6. Let (x_n) be a Cauchy sequence in a reflexive normed linear space X. Show that (x_n) converges to some point $x \in X$. **[2]**

7. Let X be a reflexive Banach space and $f \in X^*$. Prove that there exists a unit vector $x_o \in X$ such that $f(x_o) = \|f\|$. **[3]**

8. For a 2π-periodic function $f \in L^2[-\pi, \pi]$, define a sequence (φ_n) of linear functionals by

   $$\varphi_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt.$$

   Show that $\|\varphi_n\| = 1$ and $\varphi_n(f) \to 0$. **[3]**

9. Let $T : (C[0,1], \|.\|_\infty) \to (C[0,1], \|.\|_\infty)$ be a linear map defined by $Tf(t) = f\left(\frac{t}{3}\right)$. Find the spectral radius of T and show that $0 \in \sigma(T)$. Is T compact? **4**

10. Show that the space $c_o(\mathbb{N})$ is a closed and proper subspace of $l^\infty(\mathbb{N})$. **2**

11. Let φ be a bounded function on $\mathbb{R}$. Define $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $T(f)(t) = \varphi(t)f(t)$. Show that T is a bounded operator. Find the adjoint operator T^* of T. **2**

12. Let $g_n \in L^2[0,1]$ be defined by

$$g_n(t) = \begin{cases} \sqrt{n} & \text{if } 0 \leq t < 1/n, \\ 0 & \text{if } 1/n \leq t \leq 1. \end{cases}$$

Show that $\|g_n\|_2 = 1$ and g_n converges weakly to 0. **3**

13. Suppose T is a bounded linear operator on a complex Hilbert space H such that $\langle Tx, x\rangle = 0$, for all $x \in H$. Show that $T = 0$. **2**

14. Let $X = (C[0,1], \|.\|_\infty)$. Define a map $T : X \to \mathbb{C}$ by

$$T(f) = \int_0^1 tf(t)dt, \quad \text{for all } f \in X.$$

Find a vector $f_o \in X$ such that $T(f_o) = \|T\|$. **2**

15. Let T be a bounded and self-adjoint operator on a Hilbert space H. Suppose there exists $k > 0$ such that $\|Tx\| \geq k\|x\|$, for each $x \in H$. Prove that the equation $Tx = y$ has a unique solution for each $y \in H$. **3**

16. Let (T_n) be a sequence of bounded linear operator on a Banach space X such that $\|T_n - T\| \to 0$. If T_n^{-1} exist, $\forall\, n \in \mathbb{N}$ and $\|T_n^{-1}\| < 1$, then prove that $T^{-1} \in B(X)$. **4**

17. Let X be a normed linear space and Y be a Banach space. Let M be a dense subspace of X and $T \in B(M, Y)$. Show that there exists a unique extension $\tilde{T} \in B(X, Y)$ of T such that $\|\tilde{T}\| = \|T\|$. **4**

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