MA 541 (Real Analysis)

Assingment 3B

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) It is possible to define a metric d on $\mathbb{R}^2$ such that in $(\mathbb{R}^2, d)$, the sequence $\{(\frac{1}{n}, 0)\}$ converges but the sequence $\{(\frac{1}{n}, \frac{1}{n})\}$ does not converge.
   (b) There exists a metric space having exactly 36 open sets.

2. Examine whether d is a metric on X, where
   (a) $X = \mathbb{R}$ and $d(x, y) = \min\{\sqrt{|x-y|}, |x-y|^2\}$ for all $x, y \in \mathbb{R}$.
   (b) $X = \mathbb{R}$ and for all $x, y \in \mathbb{R}$, $d(x,y) = \begin{cases} 1 + |x-y| & \text{if exactly one of } x \text{ and } y \text{ is positive,} \\ |x-y| & \text{otherwise.} \end{cases}$
   (c) $X = \mathbb{R}^2$ and $d(x, y) = (|x_1 - y_1| + |x_2 - y_2|^{\frac{1}{2}})^{\frac{1}{2}}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.
   (d) $X = \mathbb{R}^n$ and $d(x, y) = [(x_1 - y_1)^2 + \frac{1}{2}(x_2 - y_2)^2 + \cdots + \frac{1}{n}(x_n - y_n)^2]^{\frac{1}{2}}$ for all $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$.
   (e) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} \min\{|z| + |w|, |z-1| + |w-1| & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$
   (f) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} |z-w| & \text{if } \frac{z}{|z|} = \frac{w}{|w|}, \\ |z| + |w| & \text{otherwise.} \end{cases}$
   (g) $X = \mathbb{C}$ and $d(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z, w \in \mathbb{C}$.

3. Examine whether $\|\cdot\|$ is a norm on $\mathbb{R}^2$, where for each $(x, y) \in \mathbb{R}^2$,
   (a) $\|(x, y)\| = (\sqrt{|x|} + \sqrt{|y|})^2$. (b) $\|(x, y)\| = \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}$.

4. Examine whether $\|\cdot\|$ is a norm on $C[0, 1]$, where for each $f \in C[0, 1]$,
   (a) $\|f\| = \min\{\|f\|_\infty, 2\|f\|_1\}$. (b) $\|f\| = \sup\{x|f(x)| : x \in [0, 1]\}$.

5. If A and B are open sets in $\mathbb{R}$ with the usual metric, then show that $A \times B$ is an open set in $\mathbb{R}^2$ with the usual (Euclidean) metric.

6. If A is a nonempty bounded subset of a metric space, then show that $\text{diam}(\overline{A}) = \text{diam}(A)$.

7. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then show that $\{m + n\alpha : m, n \in \mathbb{Z}\}$ is a dense set in $\mathbb{R}$ with the usual metric.

8. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d). Show that the sequence $(d(x_n, y_n))$ is convergent.

9. Examine whether the following metric spaces are complete.
   (a) $(\mathbb{R}, d)$, where $d(x, y) = |e^x - e^y|$ for all $x, y \in \mathbb{R}$
   (b) $((-1, 1), d)$, where $d(x, y) = |\tan\frac{\pi x}{2} - \tan\frac{\pi y}{2}|$ for all $x, y \in (-1, 1)$

10. Let X, Y be metric spaces and let $f : X \to Y$. Show that f is continuous iff $f^{-1}(B^0) \subset (f^{-1}(B))^0$ for all $B \subset Y$.
    Also, show that the continuity of f need not give the equality in the above inclusion for some

$B \subset Y$.

11. Let E be a nonempty bounded subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is uniformly continuous, then show that f is bounded on E.

12. Let (X, d_1) be a metric space and let $\lambda > 0$. Consider the metrics d_2, d_3 and d_4 on X defined by $d_2(x, y) = \lambda d_1(x, y)$, $d_3(x, y) = \min\{1, d_1(x, y)\}$ and $d_4(x, y) = \frac{d_1(x,y)}{1+d_1(x,y)}$ for all $x, y \in X$.
    (a) If (X, d_i) is a complete metric space for some $i \in \{1, 2, 3, 4\}$, then show that (X, d_j) is a complete metric space for each $j \in \{1, 2, 3, 4\}$.
    (b) If G is an open set in (X, d_i) for some $i \in \{1, 2, 3, 4\}$, then show that G is open in (X, d_j) for each $j \in \{1, 2, 3, 4\}$.
    (From (b), it follows that (X, d_i) is compact for some $i \in \{1, 2, 3, 4\}$ iff (X, d_j) is compact for each $j \in \{1, 2, 3, 4\}$.)

13. Let d be the usual metric on $\mathbb{R} \setminus \mathbb{Q}$. Show that there exists a metric ρ on $\mathbb{R} \setminus \mathbb{Q}$ such that $(\mathbb{R} \setminus \mathbb{Q}, \rho)$ is a complete metric space and for every $A \subset \mathbb{R} \setminus \mathbb{Q}$, A is open in $(\mathbb{R} \setminus \mathbb{Q}, \rho)$ iff A is open in $(\mathbb{R} \setminus \mathbb{Q}, d)$.

14. Let d be the usual metric on $\mathbb{Q}$. Does there exist a metric ρ on $\mathbb{Q}$ such that $(\mathbb{Q}, \rho)$ is a complete metric space and for every $A \subset \mathbb{Q}$, A is open in $(\mathbb{Q}, \rho)$ iff A is open in $(\mathbb{Q}, d)$.

15. Let $\{K_i : i \in I\}$ be a nonempty class of compact sets in a metric space X. Show that $\bigcap_{i \in I} K_i$ is a compact set in X and if I is finite, then $\bigcup_{i \in I} K_i$ is also a compact set in X.
    Show also that if I is infinite, then $\bigcup_{i \in I} K_i$ need not be a compact set in X.