MA 541 (Real Analysis)

Assignment 3A

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1. Examine whether d is a metric on X, where
   (a) $X = \mathbb{R}$ and $d(x, y) = |x - y|^p$ for all $x, y \in \mathbb{R}$ $(0 < p < 1)$.
   (b) $X =$ The class of all finite subsets of a nonempty set and $d(A, B) =$ The number of elements of the set $A \triangle B$ for all $A, B \in X$.

2. Let A be a closed set in a metric space X and $x \in X \setminus A$. Show that there exist disjoint open sets G and H in X such that $x \in G$ and $A \subset H$.

3. Let A and B be disjoint closed sets in a metric space X. Show that there exist disjoint open sets G and H in X such that $A \subset G$ and $B \subset H$.

4. Let X be a metric space. Show that
   (a) every closed set in X is a countable intersection of open sets in X.
   (b) every open set in X is a countable union of closed sets in X.

5. If A is a subset of a metric space X, then show that $(X \setminus A)^0 = X \setminus \overline{A}$.
   (Other similar 'commutative relations are true for closure, interior and complement. For example, $X \setminus A^0 = \overline{X \setminus A}$.)

6. Let F be a closed set in a metric space X. Prove that F is nowhere dense in X iff $X \setminus F$ is dense in X.
   Is this result true for an arbitrary subset F of X? Justify.

7. Examine whether the following metric spaces are complete.
   (a) $(\mathbb{R}, d)$, where $d(x, y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$ for all $x, y \in \mathbb{R}$
   (b) $([0, 1), d)$, where $d(x, y) = |\frac{x}{1-x} - \frac{y}{1-y}|$ for all $x, y \in [0, 1)$

8. Let X, Y be metric spaces and let $f, g : X \to Y$ be continuous. If A is a dense set in X such that $f(a) = g(a)$ for all $a \in A$, then show that $f(x) = g(x)$ for all $x \in X$.

9. Let A and B be disjoint closed sets in a metric space X. Show that there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 1$ for all $x \in A$ and $f(x) = 0$ for all $x \in B$.

10. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction and $g(\mathbf{x}) = \mathbf{x} - f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that $g : \mathbb{R}^n \to \mathbb{R}^n$ is one-one and onto. Also, show that both g and $g^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

11. Let (X, d) be a complete metric space and $f : X \to X$ be such that $f^m : X \to X$ is a contraction for some $m \in \mathbb{N}$. Show that f has a unique fixed point in X.

12. Let A and B be nonempty subsets of a metric space X.
    (a) If A and B are closed in X and $A \cap B = \emptyset$, then is it necessary that $d(A, B) > 0$? Justify.
    (b) If A is compact in X and B is closed in X, then show that there exists $a \in A$ such that $d(A, B) = d(a, B)$.

(c) If A and B are compact in X, then show that there exist $a \in A$ and $b \in B$ such that $d(A, B) = d(a, b)$.

13. Let (X, d) be a compact metric space and let $f : X \to X$ be such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Prove that $f : X \to X$ is onto.
Show also that the compactness of (X, d) is, in general, necessary in the above result.

14. Let d be the usual metric on $(0, 1)$. Show that there exists a metric ρ on $(0, 1)$ such that $((0, 1), \rho)$ is a complete metric space and for every $A \subset (0, 1)$, A is open in $((0, 1), \rho)$ iff A is open in $((0, 1), d)$.

15. Let d be the usual metric on $[0, 1]$. Does there exist a metric ρ on $[0, 1]$ such that $([0, 1], \rho)$ is a complete metric space and for every $A \subset [0, 1]$, A is open in $([0, 1], \rho)$ iff A is open in $([0, 1], d)$? Justify.

16. Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Show that there cannot exist a sequence (f_n) of real-valued continuous functions on $\mathbb{R}$ such that $f_n \to f$ pointwise on $\mathbb{R}$.