MA 541 (Real Analysis)

Assignment - 2B

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and Ω is a bounded subset of $\mathbb{R}^2$, then $f(\Omega)$ must be a bounded subset of $\mathbb{R}$.
   (b) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that all the directional derivatives of f at $(0,0)$ exist (in $\mathbb{R}$), then f must be differentiable at $(0,0)$.
   (c) There exists a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is differentiable only at $(1,0)$.
   (d) If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with $f(0,0) = (1,1)$ and $[f'(0,0)] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then there cannot exist a differentiable function $g : \mathbb{R}^2 \to \mathbb{R}^2$ with $g(1,1) = (0,0)$ and $(f \circ g)(x,y) = (y,x)$ for all $(x,y) \in \mathbb{R}^2$.
   (e) A differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ cannot have a differentiable inverse $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ if $\det[f'(x,y)] = 0$ for some $(x,y) \in \mathbb{R}^2$.
   (f) It is possible to define a metric d on $\mathbb{R}^2$ such that in $(\mathbb{R}^2, d)$, the sequence $\{(\frac{1}{n}, 0)\}$ converges but the sequence $\{(\frac{1}{n}, \frac{1}{n})\}$ does not converge.

2. Let $(\mathbf{x}_k)$ be a sequence in $\mathbb{R}^n$. Show that $(\mathbf{x}_k)$ converges in $\mathbb{R}^n$ iff for each $\mathbf{x} \in \mathbb{R}^n$, the sequence $\{\langle \mathbf{x}_k, \mathbf{x} \rangle\}$ converges in $\mathbb{R}$.

3. Examine whether the following limits exist (in $\mathbb{R}$) and find their values if they exist (in $\mathbb{R}$).

$$(a) \lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4 + y^2} \qquad (b) \lim_{(x,y)\to(0,0)} \frac{x^3 + y^2}{x^2 + y} \qquad (c) \lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1} - 1}{x^2 + y^2}$$

$$(d) \lim_{(x,y)\to(0,0)} \frac{x^3y^2 + y^6}{x^6 + y^4} \qquad (e) \lim_{(x,y)\to(0,0)} \frac{1 - \cos(x^2 + y^2)}{(x^2 + y^2)^2}$$

4. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,
   (a) $f(x,y) = \begin{cases} xy & \text{if } xy \geq 0, \\ -xy & \text{if } xy < 0. \end{cases}$ (b) $f(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$

5. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, if for all $(x,y) \in \mathbb{R}^2$,
   (a) $f(x,y) = \begin{cases} \frac{xy}{x-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$ (b) $f(x,y) = \begin{cases} xy & \text{if } xy \in \mathbb{Q}, \\ -xy & \text{if } xy \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $f(x,y) = x^2 + y^2$ for all $x \in \mathbb{Q}$, $y \in \mathbb{R} \setminus \mathbb{Q}$. Determine $f(\sqrt{2}, 2)$.

7. Let α, β be non-negative real numbers and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
   $f(x,y) = \begin{cases} \frac{|x|^\alpha |y|^\beta}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   Show that f is continuous iff $\alpha + \beta > 1$.

8. Let Ω be an open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$ and $g : \Omega \to \mathbb{R}^m$ be continuous at $\mathbf{x}_0 \in \Omega$. If for each $\varepsilon > 0$, there exist $\mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}_0)$ such that $f(\mathbf{x}) = g(\mathbf{y})$, then show that $f(\mathbf{x}_0) = g(\mathbf{x}_0)$.

9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and $\lim_{\|\mathbf{x}\|_2 \to \infty} f(\mathbf{x}) = 1$. Show that f is bounded on $\mathbb{R}^n$.

10. Examine the differentiability of f at $\mathbf{0}$, where
   (a) $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $|f(\mathbf{x})| \leq \|\mathbf{x}\|_2^2$ for all $\mathbf{x} \in \mathbb{R}^n$.
   (b) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \sqrt{|xy|}$ for all $(x, y) \in \mathbb{R}^2$.
   (c) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = ||x| - |y|| - |x| - |y|$ for all $(x, y) \in \mathbb{R}^2$.
   (d) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2 + y^2} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$
   (e) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
   (f) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$
   (g) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} \frac{\sin(x^2y^2)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
   (h) $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x, y) = \begin{cases} (\sin^2 x + x^2 \sin \frac{1}{x}, y^2) & \text{if } x \neq 0, \\ (0, y^2) & \text{if } x = 0. \end{cases}$
   (i) $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|_2\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

11. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, if for all $(x, y) \in \mathbb{R}^2$,
   (a) $f(x, y) = \begin{cases} x^2 + y^2 & \text{if both } x, y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$
   (b) $f(x, y) = \begin{cases} x^{4/3} \sin(\frac{y}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

12. Let Ω be a nonempty open subset of $\mathbb{R}^n$ and $g : \Omega \to \mathbb{R}^n$ be continuous at $\mathbf{x}_0 \in \Omega$. If $f : \Omega \to \mathbb{R}$ is such that $f(\mathbf{x}) - f(\mathbf{x}_0) = \langle g(\mathbf{x}), \mathbf{x} - \mathbf{x}_0 \rangle$ for all $\mathbf{x} \in \Omega$, then show that f is differentiable at $\mathbf{x}_0$.

13. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in \Omega$, $f(\mathbf{x}_0) = 0$ and $g : \Omega \to \mathbb{R}$ be continuous at $\mathbf{x}_0$. Prove that $fg : \Omega \to \mathbb{R}$, defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x} \in \Omega$, is differentiable at $\mathbf{x}_0$.

14. Find all $\mathbf{v} \in \mathbb{R}^2$ for which the directional derivative $f'_{\mathbf{v}}(0, 0)$ exists, where for all $(x, y) \in \mathbb{R}^2$,
   (a) $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
   (b) $f(x, y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$
   (c) $f(x, y) = ||x| - |y|| - |x| - |y|$.

15. Prove that a differentiable function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^m$ is homogeneous of degree $\alpha \in \mathbb{R}$ (*i.e.* $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and for all $t > 0$) iff $f'(\mathbf{x})(\mathbf{x}) = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

16. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that $f_x(a, b) = f_y(a, b)$ for all $(a, b) \in \mathbb{R}^2$ and $f(a, 0) > 0$ for all $a \in \mathbb{R}$. Show that $f(a, b) > 0$ for all $(a, b) \in \mathbb{R}^2$.

17. Let Ω be an open subset of $\mathbb{R}^n$ such that $\mathbf{a}, \mathbf{b} \in \Omega$ and $S = \{(1 - t)\mathbf{a} + t\mathbf{b} : t \in [0, 1]\} \subset \Omega$. If $f : \Omega \to \mathbb{R}^m$ is differentiable at each point of S, then show that there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(\mathbf{b}) - f(\mathbf{a}) = L(\mathbf{b} - \mathbf{a})$.

18. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable such that $\langle f'(\mathbf{x})(\mathbf{y}), \mathbf{y} \rangle \geq \|\mathbf{y}\|_2^2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that $\|f(\mathbf{x}) - f(\mathbf{y})\|_2 \geq \|\mathbf{x} - \mathbf{y}\|_2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

19. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is locally invertible, if for all $(x, y) \in \mathbb{R}^2$,
(a) $f(x, y) = (x^2 + y^2, xy)$. (b) $f(x, y) = (\cos x + \cos y, \sin x + \sin y)$.

20. Determine all the points of $\mathbb{R}^3$ where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is locally invertible, if for all $(x, y, z) \in \mathbb{R}^3$,
(a) $f(x, y, z) = (x + y, xy + z, y + z)$. (b) $f(x, y, z) = (x - xy, xy - xyz, xyz)$.

21. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $\Omega' = \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 : x \leq 0\}$. Show that the function $f : \Omega \to \Omega'$, defined by $f(x, y) = (x^2 - y^2, 2xy)$ for all $(x, y) \in \Omega$, is differentiable and invertible. Is $f^{-1} : \Omega' \to \Omega$ differentiable? Justify.

22. Let $f(x, y) = (3x - y^2, 2x + y, xy + y^3)$ and $g(x, y) = (2ye^{2x}, xe^y)$ for all $(x, y) \in \mathbb{R}^2$. Examine whether $(f \circ g^{-1})'(2, 0)$ exists (with a meaningful interpretation of g^{-1}) and find $(f \circ g^{-1})'(2, 0)$ if it exists.

23. Show that there are points $(x, y, z, u, v, w) \in \mathbb{R}^6$ which satisfy the equations

$$x^2 + u + e^v = 0,$$
$$y^2 + v + e^w = 0,$$
$$z^2 + w + e^u = 0.$$

Prove that in a neighbourhood of such a point there exist unique differentiable solutions $u = \varphi_1(x, y, z)$, $v = \varphi_2(x, y, z)$, $w = \varphi_3(x, y, z)$. If $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, find $\varphi'(x, y, z)$.

24. Show that the system of equations

$$x^2 + y^2 - u^2 - v = 0,$$
$$x^2 + 2y^2 + 3u^2 + 4v^2 = 1,$$

defines (u, v) implicitly as a differentiable function of (x, y) locally around the point $(x, y, u, v) = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ but does not define (x, y) implicitly as a differentiable function of (u, v) locally around the same point.

25. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \frac{x^2 y(x-y)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
Examine whether $f_{xy}(0, 0) = f_{yx}(0, 0)$.

26. Find the 3rd order Taylor polynomial of $f(x, y, z) = x^2 y + z$ about the point $(1, 2, 1)$.

27. Find the 4th order Taylor polynomial of $g(x, y) = e^{x-2y}/(1 + x^2 - y)$ about the point $(0, 0)$.