MA 541 (Real Analysis)

Assignment 2A

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) A continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ cannot be one-one and onto if $\det[f'(x, y)] = 0$ for some $(x, y) \in \mathbb{R}^2$.

2. Examine the differentiability of f at $\mathbf{0}$, where
   (a) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} x & \text{if } |x| < |y|, \\ -x & \text{if } |x| \geq |y|. \end{cases}$
   (b) $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$.

3. If $f(x, y) = |x| \sin(x^2 + y^2)$ for all $(x, y) \in \mathbb{R}^2$, then determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable and $\mathbf{y} \in \mathbb{R}^m$. If $g(\mathbf{x}) = \langle f(\mathbf{x}), \mathbf{y} \rangle$ for all $\mathbf{x} \in \mathbb{R}^n$, then show that $g : \mathbb{R}^n \to \mathbb{R}$ is differentiable.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable such that $f(1, 1) = 1$, $f_x(1, 1) = 2$ and $f_y(1, 1) = 5$. If $g(x) = f(x, f(x, x))$ for all $x \in \mathbb{R}$, determine $g'(1)$.

6. Let Ω be an open convex subset of $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function and $f : \Omega \to \mathbb{R}^m$ be differentiable with $f'(\mathbf{x}) = L$ for all $\mathbf{x} \in \Omega$. Show that there exists $\mathbf{v} \in \mathbb{R}^m$ such that $f(\mathbf{x}) = L(\mathbf{x}) + \mathbf{v}$ for all $\mathbf{x} \in \Omega$.

7. For $n \geq 2$, let $B = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < 1\}$ and $f(\mathbf{x}) = \|\mathbf{x}\|_2^2 \mathbf{x}$ for all $\mathbf{x} \in B$. Show that $f : B \to B$ is differentiable and invertible but that $f^{-1} : B \to B$ is not differentiable at $\mathbf{0}$.

8. Using implicit function theorem, show that in a neighbourhood of any point $(x_0, y_0, u_0, v_0) \in \mathbb{R}^4$ which satisfies the equations
$$x - e^u \cos v = 0,$$
$$v - e^y \sin x = 0,$$
   there exists a unique solution $(u, v) = \varphi(x, y)$ satisfying $\det[\varphi'(x, y)] = v/x$.

9. Show that around the point $(0, 1, 1)$, the equation $xy - z \log y + e^{xz} = 1$ can be solved locally as $y = f(x, z)$ but cannot be solved locally as $z = g(x, y)$.