MA 541 (Real Analysis)

Assignment 1B

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If both (x_n) and (y_n) are unbounded sequences of positive real numbers, then the sequence $(x_n y_n)$ cannot be convergent.
   (b) A monotonic sequence (x_n) in $\mathbb{R}$ is convergent iff the sequence (x_n^2) is convergent.
   (c) If an increasing sequence (x_n) in $\mathbb{R}$ has a convergent subsequence, then (x_n) must be convergent.
   (d) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) < 3$ for all $x \in \mathbb{Q}$, then it is necessary that $f(x) < 3$ for all $x \in \mathbb{R}$.
   (e) There exists a continuous function from $(0, 1]$ onto $\mathbb{R}$.
   (f) If $f : \mathbb{R} \to \mathbb{R}$ is continuous at both 2 and 4, then f must be continuous at some $c \in (2, 4)$.
   (g) If $f : [1, 2] \to \mathbb{R}$ is differentiable, then f' must be bounded on $[1, 2]$.
   (h) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then for each $c \in \mathbb{R}$, there must exist $a, b \in \mathbb{R}$ with $a < c < b$ such that $f(b) - f(a) = (b - a)f'(c)$.
   (i) The function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x + \sin x$ for all $x \in \mathbb{R}$, is strictly increasing on $\mathbb{R}$.

2. Find $\sup A$ and $\inf A$, where (a) $A = \{\frac{m}{m+n} : m, n \in \mathbb{N}\}$ (b) $A = \{\frac{m}{|m|+n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$.

3. Let A, B be nonempty subsets of $\mathbb{R}$.
   (a) If A and B are bounded above, then show that $A + B = \{a + b : a \in A, b \in B\}$ is bounded above and that $\sup(A + B) = \sup A + \sup B$.
   (b) If A and B are bounded below, then show that $A+B$ is bounded below and that $\inf(A+B) = \inf A + \inf B$.

4. Let A be a nonempty bounded subset of $\mathbb{R}$ and let $\alpha \in \mathbb{R}$. If $\alpha A = \{\alpha a : a \in A\}$, then show that $\sup(\alpha A) = \begin{cases} \alpha \sup A & \text{if } \alpha \geq 0, \\ \alpha \inf A & \text{if } \alpha < 0, \end{cases}$ and $\inf(\alpha A) = \begin{cases} \alpha \inf A & \text{if } \alpha \geq 0, \\ \alpha \sup A & \text{if } \alpha < 0. \end{cases}$

5. Let A, B be nonempty subsets of $(0, \infty)$. If A and B are bounded above, then show that $AB = \{ab : a \in A, b \in B\}$ is bounded above and that $\sup(AB) = \sup A \cdot \sup B$.

6. Let (x_n) be a convergent sequence in $\mathbb{R}$ with limit $\ell \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$.
   (a) If $x_n > \alpha$ for all $n \in \mathbb{N}$, then show that $\ell \geq \alpha$.
   (b) If $\ell > \alpha$, then show that there exists $n_0 \in \mathbb{N}$ such that $x_n > \alpha$ for all $n \geq n_0$.
   (Note that ℓ can be equal to α in (a).)

7. If $x_n = (1 + \frac{1}{n})^n$ and $y_n = (1 + \frac{1}{n})^{n+1}$ for all $n \in \mathbb{N}$, then show that the sequence (x_n) is increasing and bounded above whereas the sequence (y_n) is decreasing and bounded below. (Thus both (x_n) and (y_n) are convergent.)

8. Let a sequence (x_n) in $\mathbb{R}$ satisfy either of the following conditions:
   (a) There exists $\alpha \in (0, 1)$ such that $|x_{n+1} - x_n| \leq \alpha^n$ for all $n \in \mathbb{N}$.
   (b) There exists $\alpha \in (0, 1)$ such that $|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$.

Show that (x_n) is a Cauchy sequence (and hence convergent).

9. Examine whether the sequences (x_n) defined as below are convergent. Also, find their limits if they are convergent.
(a) $x_n = (a^n + b^n + c^n)^{\frac{1}{n}}$ for all $n \in \mathbb{N}$, where a, b, c are distinct positive real numbers.
(b) $x_n = \frac{1}{n^2}(a_1 + \cdots + a_n)$, where $a_n = n + \frac{1}{n}$ for all $n \in \mathbb{N}$.
(c) $x_n = \frac{1}{n^2}([\alpha] + [2\alpha] + \cdots + [n\alpha])$, where $\alpha \in \mathbb{R}$.
(d) $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n+1}}$ for all $n \in \mathbb{N}$.
(e) $x_n = (\frac{\sin n + \cos n}{3})^n$ for all $n \in \mathbb{N}$.
(f) $x_n = \frac{n}{3} - [\frac{n}{3}]$ for all $n \in \mathbb{N}$.
(g) $x_n = (n^2+1)^{\frac{1}{8}} - (n+1)^{\frac{1}{4}}$ for all $n \in \mathbb{N}$.
(h) $x_1 = 4$ and $x_{n+1} = 3 - \frac{2}{x_n}$ for all $n \in \mathbb{N}$.
(i) $x_1 = 1$ and $x_{n+1} = (\frac{n}{n+1})x_n^2$ for all $n \in \mathbb{N}$.
(j) $x_1 = 0$ and $x_{n+1} = \sqrt{6 + x_n}$ for all $n \in \mathbb{N}$.
(k) $x_1 = 1$ and $x_{n+1} = \frac{2+x_n}{1+x_n}$ for all $n \in \mathbb{N}$.
(l) $x_1 = a$, $x_2 = b$ and $x_{n+2} = \frac{1}{2}(x_n + x_{n+1})$ for all $n \in \mathbb{N}$, where $a, b \in \mathbb{R}$.

10. Let $a > 0$ and let $x_1 = 0$, $x_{n+1} = x_n^2 + a$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) is convergent iff $a \le \frac{1}{4}$.

11. Let (x_n) be a sequence in $\mathbb{R}$ such that each of the subsequences (x_{2n}), (x_{2n-1}) and (x_{3n}) converges. Show that (x_n) is convergent.

12. Let (x_n) be a sequence in $\mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = 0$. Show that there exists a subsequence (x_{n_k}) of (x_n) such that the series $\sum\limits_{k=1}^{\infty} x_{n_k}$ is absolutely convergent.

13. Let (x_n) and (y_n) be bounded sequences in $\mathbb{R}$. Show that
(a) $\limsup\limits_{n\to\infty}(x_n + y_n) \le \limsup\limits_{n\to\infty} x_n + \limsup\limits_{n\to\infty} y_n$.
(b) $\liminf\limits_{n\to\infty}(x_n + y_n) \ge \liminf\limits_{n\to\infty} x_n + \liminf\limits_{n\to\infty} y_n$.
Give examples to show that the inequalities in (a) and (b) can be strict.
Also, show that if either (x_n) or (y_n) is convergent, then the equality holds in both (a) and (b).

14. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that for each $x \in \mathbb{Q}$, $f(x)$ is an integer. If $f(\frac{1}{2}) = 2$, then find $f(\frac{1}{3})$.

15. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Show that f is a constant function.

16. Let $f : (0, \infty) \to \mathbb{R}$ be continuous such that $\lim\limits_{x\to 0+} f(x) = 0$ and $\lim\limits_{x\to\infty} f(x) = 1$. Show that there exists $c \in (0, \infty)$ such that $f(c) = \frac{\sqrt{3}}{2}$.

17. Let $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ be continuous such that $\sup\{f(x) : x \in [0, 1]\} = \sup\{g(x) : x \in [0, 1]\}$. Show that there exists $c \in [0, 1]$ such that $f(c) = g(c)$.

18. Let $f : [a, b] \to \mathbb{R}$ be continuous. For $n \in \mathbb{N}$, let $x_1, ..., x_n \in [a, b]$ and let $\alpha_1, ..., \alpha_n$ be nonzero real numbers having same sign. Show that there exists $c \in [a, b]$ such that
$f(c) \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \alpha_i f(x_i)$.
(In particular, this shows that if $f : [a, b] \to \mathbb{R}$ is continuous and if for $n \in \mathbb{N}$, $x_1, ..., x_n \in [a, b]$, then there exists $\xi \in [a, b]$ such that $f(\xi) = \frac{1}{n}(f(x_1) + \cdots + f(x_n))$.)

19. Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(a) = f(b)$. Show that for each $\varepsilon > 0$, there exist distinct $x, y \in [a, b]$ such that $|x - y| < \varepsilon$ and $f(x) = f(y)$.

20. Let p be a non-constant polynomial of even degree with real coefficients in one real variable. Prove that exactly one of the following two statements holds.
(a) There exists $x_0 \in \mathbb{R}$ such that $p(x_0) \leq p(x)$ for all $x \in \mathbb{R}$.
(b) There exists $x_0 \in \mathbb{R}$ such that $p(x_0) \geq p(x)$ for all $x \in \mathbb{R}$.

21. Let $f(x) = |x^3|$ for all $x \in \mathbb{R}$. Examine the existence of $f'(x)$, $f''(x)$ and $f'''(x)$, where $x \in \mathbb{R}$.

22. Examine whether $f : \mathbb{R} \to \mathbb{R}$, defined as below, is differentiable at 0.
(a) $f(x) = \begin{cases} \frac{1}{2^{n+1}} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$
(b) $f(x) = \begin{cases} \frac{1}{4^n} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

23. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at x_0 and let $g(x) = |f(x)|$ for all $x \in \mathbb{R}$. Show that $g : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 iff either $f(x_0) \neq 0$ or $f(x_0) = f'(x_0) = 0$.

24. Let $f : [a, b] \to \mathbb{R}$ be differentiable such that $f(x) \neq 0$ for all $x \in [a, b]$. Show that there exists $c \in (a, b)$ such that $\frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}$.

25. Let $f : [0, 1] \to \mathbb{R}$ be differentiable such that $f(0) = 0$ and $f(1) = 1$. Show that there exist $c_1, c_2 \in [0, 1]$ with $c_1 \neq c_2$ such that $f'(c_1) + f'(c_2) = 2$.

26. Let $f : [0, 1] \to \mathbb{R}$ be differentiable such that $f(0) = f(1) = 0$. Show that there exists $c \in (0, 1)$ such that $f'(c) = f(c)$.

27. Determine all the differentiable functions $f : [0, 1] \to \mathbb{R}$ satisfying the conditions
(a) $f(0) = 0$, $f(1) = 1$ and $|f'(x)| \leq \frac{1}{2}$ for all $x \in [0, 1]$.
(b) $f(0) = 0$, $f(1) = 1$ and $|f'(x)| \leq 1$ for all $x \in [0, 1]$.

28. Show that for each $a \in (0, 1)$ and for each $b \in \mathbb{R}$, the equation $a \sin x + b = x$ has a unique root in $\mathbb{R}$.

29. Show that the equation $|x^{10} - 60x^9 - 290| = e^x$ has at least one real root.

30. Show that for each $n \in \mathbb{N}$, the equation $x^n + x - 1 = 0$ has a unique root in $[0, 1]$.
If for each $n \in \mathbb{N}$, x_n denotes this root, then show that the sequence (x_n) converges to 1.

31. Prove that for each $a \geq 0$, there exists a unique $b \geq 0$ such that $a = \int\limits_0^b \frac{1}{(1+x^3)^{1/5}}\, dx$.

32. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(-1) = 5$, $f(0) = 0$ and $f(1) = 10$. Show that there exist $c_1,\ c_2 \in (-1, 1)$ such that $f'(c_1) = -3$ and $f'(c_2) = 3$.

33. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f''(c)$ exists (in $\mathbb{R}$), where $c \in \mathbb{R}$. Show that
$\lim\limits_{h\to 0} \frac{f(c+h)-2f(c)+f(c-h)}{h^2} = f''(c)$.
Give an example of an $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$ for which $f''(c)$ does not exist (in $\mathbb{R}$) but the above limit exists (in $\mathbb{R}$).

34. Using Taylor's theorem, show that
(a) $|\sqrt{1+x} - (1 + \frac{x}{2} - \frac{x^2}{8})| \leq \frac{1}{2}|x|^3$ for all $x \in (-\frac{1}{2}, \frac{1}{2})$.
(b) $x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for all $x \in (0, \pi)$.

35. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable such that f, f' and f'' are bounded on $\mathbb{R}$. Show that $\sup\limits_{x\in\mathbb{R}} |f'(x)|^2 \leq 4 \sup\limits_{x\in\mathbb{R}} |f(x)| \cdot \sup\limits_{x\in\mathbb{R}} |f''(x)|$.

36. Let $f : [0, 1] \to \mathbb{R}$ be continuous and $f(1) = 0$. If $f_n(x) = f(x)x^n$ for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$, then examine whether the sequence (f_n) converges uniformly on $[0, 1]$.

37. Let $f_n(x) = nx(1-x^2)^n$ for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$. Examine the pointwise and uniform convergence of the sequence (f_n) on $[0, 1]$.
Also, examine the validity of the equality $\lim\limits_{n\to\infty} \int\limits_0^1 f_n(x)\, dx = \int\limits_0^1 (\lim\limits_{n\to\infty} f_n(x))\, dx$.

38. Let $E(\neq \emptyset) \subset \mathbb{R}$ and let $f, g, f_n, g_n : E \to \mathbb{R}$ $(n \in \mathbb{N})$ be such that $f_n \to f$ uniformly on E and $g_n \to g$ uniformly on E.
(a) Show that $f_n + g_n \to f + g$ uniformly on E.
(b) Is it necessary that $f_n.g_n \to f.g$ uniformly on E? Justify.
(c) If f and g are bounded on E, then show that $f_n.g_n \to f.g$ uniformly on E.

39. Let (f_n) be a sequence of real-valued uniformly continuous functions on a nonempty set $E \subset \mathbb{R}$. If $f : E \to \mathbb{R}$ is such that $f_n \to f$ uniformly on E, then show that f is uniformly continuous on E.
Does this result hold if $f_n \to f$ pointwise on E? Justify.

40. Let $E(\neq \emptyset) \subset \mathbb{R}$ and let $f, f_n : E \to \mathbb{R}$ $(n \in \mathbb{N})$ be such that $f_n \to f$ uniformly on E. If $g : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then show that $g \circ f_n \to g \circ f$ uniformly on E.