MA 541 (Real Analysis)

Assignment 1A

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$ and $f(x) \in \mathbb{R} \setminus \mathbb{Q}$ for all $x \in \mathbb{Q}$.
   (b) If (f_n) is a sequence in $C[0,1]$ such that $|f_{n+1}(x) - f_n(x)| \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0,1]$, then there must exist $f \in C[0,1]$ such that $\int_0^1 |f_n(x) - f(x)|\, dx \to 0$ as $n \to \infty$.

2. Let A be a nonempty bounded subset of $\mathbb{R}$. Show that $\sup\{|x-y| : x, y \in A\} = \sup A - \inf A$.

3. Let (x_n) be a convergent sequence of positive real numbers such that $\lim\limits_{n\to\infty} x_n < 1$. Show that $\lim\limits_{n\to\infty} x_n^n = 0$.

4. Let (x_n) be a sequence in $\mathbb{R}$ and let $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$ for all $n \in \mathbb{N}$. If (x_n) is convergent, then show that (y_n) is also convergent.
   If (y_n) is convergent, then is it necessary that (x_n) is (a) convergent? (b) bounded? Justify.

5. For $a \in \mathbb{R}$, let $x_1 = a$ and $x_{n+1} = \frac{1}{4}(x_n^2 + 3)$ for all $n \in \mathbb{N}$. Examine the convergence of the sequence (x_n) for different values of a. Also, find $\lim\limits_{n\to\infty} x_n$ whenever it exists (in $\mathbb{R}$).

6. Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If every subsequence of (x_n) has a further subsequence converging to x, then show that $x_n \to x$.

7. Let (x_n) be a sequence of nonzero real numbers. Prove or disprove the following statements.
   (a) If (x_n) is unbounded, then the sequence $(\frac{1}{x_n})$ must converge to 0.
   (b) If (x_n) does not have any convergent subsequence, then the sequence $(\frac{1}{x_n})$ must converge to 0.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ [x] & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
   Determine all the points of $\mathbb{R}$ where f is continuous.

9. Let $f : [0,1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Show that
   (a) there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{2}$.
   (b) there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{3}$.

   (In fact, if $n \in \mathbb{N}$, then there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{n}$. However, it is not necessary that there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{2}{5}$.)

10. Let p be an odd degree polynomial with real coefficients in one real variable. If $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

    (In particular, this shows that

(a) every odd degree polynomial with real coefficients in one real variable has at least one real zero.

(b) the equation $x^9 - 4x^6 + x^5 + \frac{1}{1+x^2} = \sin 3x + 17$ has at least one real root.

(c) the range of every odd degree polynomial with real coefficients in one real variable is $\mathbb{R}$.)

11. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
    (a) Is it possible for f to be not continuous? Justify.
    (b) If f is continuous at some point of $\mathbb{R}$, then show that $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 |\cos \frac{\pi}{x}| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
    Examine whether f is differentiable (a) at 0 (b) on $(0, 1)$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable at 0. If $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$, then find $f'(0)$ and $f''(0)$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(0) = f(1) = 0$ and $f'(0) > 0$, $f'(1) > 0$. Show that there exist $c_1, c_2 \in (0, 1)$ with $c_1 \neq c_2$ such that $f'(c_1) = f'(c_2) = 0$.

15. For $n \in \mathbb{N}$, show that the equation $1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n} = 0$ has exactly one real root if n is odd and has no real root if n is even.

16. Let $A(\neq \emptyset) \subset \mathbb{R}^n$ be such that every continuous function $f : A \to \mathbb{R}$ is bounded. Show that A is a closed and bounded subset of $\mathbb{R}^n$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lim\limits_{|x| \to \infty} f(x) = 0$. Show that f is uniformly continuous on $\mathbb{R}$.

18. Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
    Show that there cannot exist a sequence (f_n) of real-valued continuous functions on $\mathbb{R}$ such that $f_n \to f$ pointwise on $\mathbb{R}$.

19. Let $f_n(x) = nx(1 - x^2)^n$ for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$. Examine the pointwise and uniform convergence of the sequence (f_n) on $[0, 1]$.
    Also, examine the validity of the equality $\lim\limits_{n \to \infty} \int_0^1 f_n(x)\, dx = \int_0^1 (\lim\limits_{n \to \infty} f_n(x))\, dx$.

20. Let $E(\neq \emptyset) \subset \mathbb{R}$ and let (f_n) be a sequence of real-valued bounded functions on E. If $f : E \to \mathbb{R}$ is such that $f_n \to f$ uniformly on E, then show that f is bounded on E.
    Does this result hold if $f_n \to f$ pointwise on E? Justify.

21. If $f(x) = \sum\limits_{n=1}^{\infty} \frac{\sin(nx^2)}{n^3+1}$ for all $x \in \mathbb{R}$, then show that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

22. Let $f : [0, 1] \to \mathbb{R}$ be continuous and $\int_0^1 x^n f(x)\, dx = 0$ for all $\mathbb{N} \cup \{0\}$. Show that $f(x) = 0$ for all $x \in [0, 1]$.