MA15010H: Multi-variable Calculus

(Assignment 7 Hint/Model solutions: Line and surface integrals)

September - November, 2025

1. Find the line integral of the vector field $F(x, y, z) = yi - xj + k$ along the path $c(t) = (\cos t, \sin t, \frac{t}{2\pi})$, $0 \le t \le 2\pi$ joining $(1, 0, 0)$ to $(1, 0, 1)$.
   Solution:
   $$\int F \cdot dc = \int_0^{2\pi} F((c(t)) \cdot c'(t)dt = 1 - 2\pi.$$
2. Evaluate $\int\limits_C T \cdot dR$, where C is the circle $x^2 + y^2 = 1$ and T is the unit tangent vector.
   Solution: The unit circle can be represented by $C = \{R(t) : \; 0 \le t < 2\pi\}$. The unit tangent vector T to C is given $T(t) = \frac{R'(t)}{\|R'(t)\|}$. Hence
   $$\int_C T \cdot dR = \int_0^{2\pi} \frac{R'(t)}{\|R'(t)\|} \cdot R'(t)dt = 2\pi.$$
3. Show that the integral $\int\limits_C yzdx + (xz+1)dy + xydz$ is independent of the path C joining $(1, 0, 0)$ and $(2, 1, 4)$.
   Solution: Let $F(x, y, z) = (yz, xz + 1, xy)$. Consider $f(x, y, z) = xyz + y + c$. Then $\nabla f(x, y, z) = (yz, xz + 1, xy) = F(x, y, z)$. Hence, by second FTC for line integral
   $$\int_C \nabla f \cdot dR = f(2, 1, 4) - f(1, 0, 0).$$
   That is, the given line integral is path independent. **Note that** one can $\nabla f = F$ for f by doing indefinite integral.
4. Use Green's Theorem to compute $\int\limits_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ where C is the boundary of the region $\{(x, y) : x, y \ge 0 \text{ and } x^2 + y^2 \le 1\}$.
   Solution: Let $M(x, y) = 2x^2 - y^2$ and $N = x^2 + y^2$. Then $N_x - M_y = 2(x + y)$. Let $D = \{(x, y) : x, y \ge 0 \text{ and } x^2 + y^2 \le 1\}$.

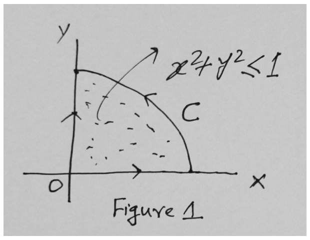


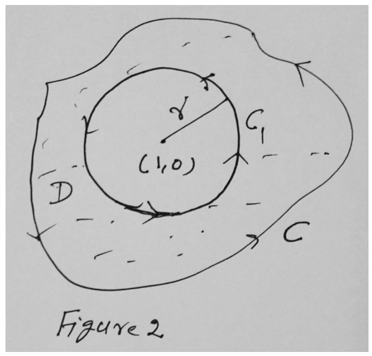
Figure 1

   Note that C is a simple closed and piece wise smooth curve, as shown in Figure 1. Then by Green's theorem
   $$\int_C Mdx + Ndy = \iint\limits_D 2(x + y)dxdy.$$

Use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, where $0 < r \le 1$ and $0 \le \theta \le \frac{\pi}{2}$.

5. If C is any simple closed and smooth curve in $\mathbb{R}^2$ which is not passing through the point $(1, 0)$, then evaluate the integral $\int\limits_C \frac{-ydx+(x-1)dy}{(x-1)^2+y^2}$.

Solution: Let $M(x, y) = -\frac{y}{(x-1)^2+y^2}$ and $N(x, y) = \frac{x-1}{(x-1)^2+y^2}$. Note that M and N are not continuous at $(0, 0)$. Let C_1 be a circle of radius r centered at $(1, 0)$ which is in the interior of domain D enclosed by C. Please see Figure 2.



A simple calculation shows that $N_x - M_y = 0$ on D. By Green'n theorem for multiply connected domain,

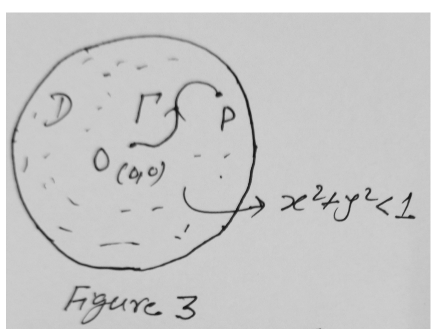
$$\int_C Mdx + Ndy - \int_{C_1} Mdx + Ndy = \iint_D (N_x - M_y)dxdy = 0.$$

Use the parametrization $x - 1 = r\cos t$ and $y = r\sin t,\ 0 \le t < 2\pi$. Then

$$\int_C Mdx + Ndy = \int_{C_1} Mdx + Ndy = 2\pi.$$

6. Let $D = \{(x, y) :\ x^2 + y^2 < 1\}$. If $f : D \to R^2$ is a continuously differentiable function such that $\int\limits_\Gamma f \cdot dR = 0$ for every curve Γ in D, then f constant.

Solution: Let $P = (x, y)$ be an arbitrary point in D. Then there exists a smooth curve Γ connecting origin and P as shown in Figure 3.

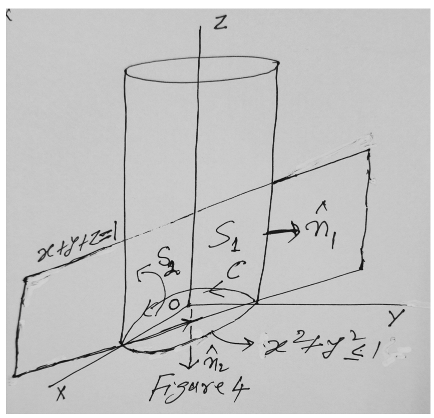


Let $\Gamma = \{R(t):\ a \le t \le s\}$ and $f = (f_1, f_2)$. Then by the given condition

$$\int_a^s f(R(t)) \cdot R'(t)dt = 0.$$

Since point P is arbitrary, the above condition holds for every choice of $s \le a$. Hence, $f(R(t)) \cdot R'(t) = 0$ for all $t \in [a, s]$. Choose $R(t) = (t + x, y)$ then $R'(s) = (1, 0)$. This implies $f_1(P) = f_1(R(s)) = 0$. Similarly, we can select $R(t)$ such that $R'(s) = (0, 1)$. Hence, $f_2(P) = 0$. Thus, $f = 0$ on D.

7. Use Stokes' Theorem to evaluate the line integral $\int_C -y^3dx + x^3dy - z^3dz$, where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ and the orientation of C corresponds to counterclockwise motion in the xy-plane.
Solution: Let $S = S_1 \cup S_2$, as shown in Figure 4. Note that S_1 is the surface of part of the cylinder whereas S_2 is the base of the cylinder.



Let $F(x, y, z) = (-y^3, x^3, -z^3)$. Then by stoke's theorem,

$$\oint_C F \cdot dR = \iint_S \text{curl } F \cdot \hat{n}\, d\sigma = \left(\iint_{S_1} + \iint_{S_2}\right) \text{curl } F \cdot \hat{n}\, d\sigma.$$

Note that curl $F(x, y, z) = 3(x^2 + y^2)k$. Unit vector $\hat{n}_1$ on S_1 is given by $\hat{n}_1 = \frac{r_\alpha \times r_\beta}{\|r_\alpha \times r_\beta\|}$, and $d\sigma_1(\alpha, \beta) = \|r_\alpha \times r_\beta\|\, d\alpha\, d\beta$, where $r(\alpha, \beta) = (\cos\alpha, \sin\frac{\alpha}{\beta})$ with $0 \le \alpha < 2\pi$, and $0 \le \beta \le 1 - \cos\alpha - \sin\alpha$. Hence

$$\iint_{S_1} \text{curl } F \cdot \hat{n}_1\, d\sigma = \int_{\alpha=0}^{2\pi} \int_{\beta=0}^{1-\cos\alpha-\sin\alpha} (3k) \cdot (r_\alpha \times r_\beta) d\alpha d\beta.$$

Further, unit vector $\hat{n}_2$ on S_2 is given by $\hat{n}_2(\alpha, \beta) = -k$ and $d\sigma_2(x, y) = dxdy$. Thus,

$$\iint_{S_2} \text{curl } F \cdot \hat{n}_2 \, d\sigma = \iint_R -3(x^2 + y^2)dxdy,$$

where R is the region $\{(x, y) : x^2 + y^2 \leq 1 \text{ and } x + y \leq 1\}$.

8. Let $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and let S be any surface that surrounds the origin. Prove that $\iint_S \vec{F} \cdot \hat{n} d\sigma = 4\pi$.

Solution: Note that $\vec{F}$ is not continuous at the origin O. Let S_ρ be the sphere of radius ρ centered at O so that S_ρ is in the interior of the domain D enclosed by S, as shown in Figure 5. A simple calculation show that $\text{div}\, F = 0$ on D.

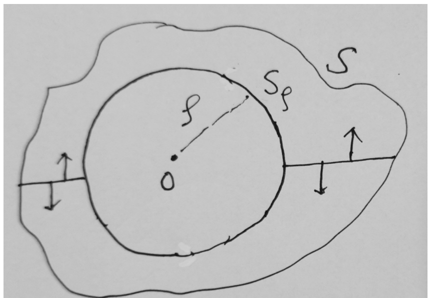


Figure 5

By divergence theorem, we get

$$\iint_S \vec{F} \cdot \hat{n} d\sigma - \iint_{S_\rho} \vec{F} \cdot \hat{n} d\sigma = \iint_D \text{div}\, \vec{F} dV = 0.$$

The surface of S_ρ can be represented by $F(x, y, z) = x^2 + y^2 + z^2 - \rho^2$. Hence the unit vector on S_ρ is given by $\hat{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{\vec{r}}{\rho}$. Note that the surface of S_ρ can be represented by $v(\theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$. Hence surface element on S_ρ will be given by $d\sigma(\theta, \phi) = \|v_\theta \times v_\phi\| d\theta d\phi = \rho^2 \sin\phi d\theta d\phi$, where $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$, $0 \leq \theta < 2\pi$ and $0 \leq \phi < \pi$. Thuis,

$$\iint_S \vec{F} \cdot \hat{n} d\sigma = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{\rho} \rho^2 \sin\phi d\theta d\phi = 4\pi.$$

9. Let D be the domain inside the cylinder $x^2 + y^2 = 1$ cut off by the planes $z = 0$ and $z = x + 2$. If $\vec{F} = (x^2 + ye^z, y^2 + ze^x, z + xe^y)$, use divergence theorem to evaluate $\iint_{\partial D} F \cdot \hat{n} \, d\sigma$.

Solution: Please refer to Figure 6.

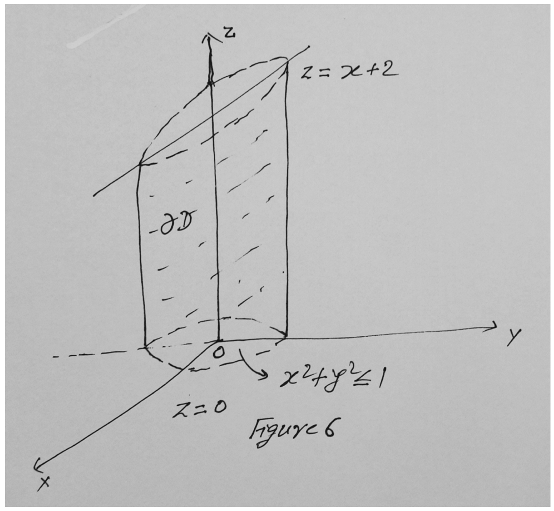


Figure 6

By divergence theorem, we get

$$\iint_{\partial D} F \cdot \hat{n}\, d\sigma = \iint_{D} \operatorname{div} \vec{F} dV = \iint_{x^2+y^2\leq 1} \left(\int_{z=0}^{x+2} (2x+2y+1)dz \right) dxdy.$$

Use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$.