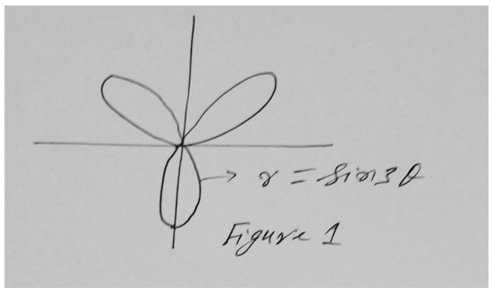
MA15010H: Multi-variable Calculus

(Assignment 6 Hint/Model solutions: Change of variables, triple integral)

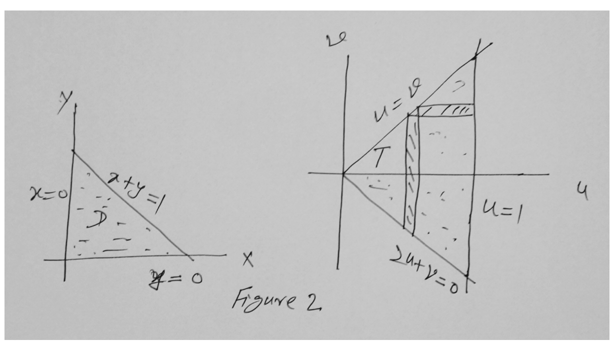
September - November, 2025

1. Using double integral, find the area enclosed by the curve $r = \sin 3\theta$ given in polar coordinates.
   Solution: Please see Figure 1.



The curve is given by $r = \sin 3\theta$, where $\theta \in [0, 2\pi)$. Area $= 3 \int\limits_{0}^{\frac{\pi}{3}} \int\limits_{r=0}^{\sin 3\theta} r dr d\theta$.

2. Evaluate the double integral $\iint\limits_{D} \sqrt{x+y}\ (y-2x)^2 dydx$ over the domain D bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.
   Solution: Let $u = x + y$ and $v = y - 2x$. Then $x = \frac{u-v}{3}$ and $y = \frac{2u+v}{3}$.



Here $J(u, v) = \frac{1}{3}$. Note that the line $y = 0$ is mapped to $u = x$ and $v = -2x$. Similarly, the line $x = 0$ is mapped to $u = y$ and $v = y$. That is, $x = 0$ is mapped to $u = v$. Also, $x + y = 1$ is mapped to $u = 1$. Interior of D mapped to the interior of the triangle T as shown in the Figure 2. Hence

$$\iint\limits_{D} \sqrt{x+y}\ (y-2x)^2 dydx = \frac{1}{3} \iint\limits_{T} \sqrt{u}\ v^2 dvdu = \frac{1}{3} \int_{u=0}^{1} \left(\int_{v=-2u}^{u} \sqrt{u}\ v^2 dv \right) du.$$

3. Evaluate the integral $\iint\limits_D e^{(x-2y)}dxdy$ over the domain D bounded by the lines $x-2y=0$, $2x-y=0$ and $x+y=1$ as shown in Figure 3.

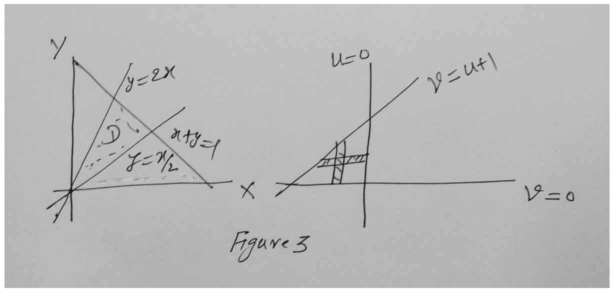


Figure 3

Solution: Put $u=x-2y$ and $v=2x-y$. Then $x=\frac{2v-u}{3}$ and $y=\frac{v-2u}{3}$. It is clear that $x-2y=0$ is mapped to $u=0$ and $2x-y=0$ is mapped to $v=0$. Also, $x+y=1$ is mapped to $v-u=1$. Here $J(u,v)=\frac{1}{3}$. Hence

$$\iint\limits_D e^{(x-2y)}dxdy = \frac{1}{3}\int_{u=-1}^{0}\left(\int_{v=0}^{u+1} e^u dv\right)du = \frac{1}{3}\int_{u=-1}^{0} e^u(u+1)du.$$

4. Compute $\lim\limits_{a\to\infty}\iint\limits_{D(a)} e^{-(x^2+y^2)}dxdy$, where

$(a)\ \ D(a)=\{(x,y): x^2+y^2\le a^2\}$ and $(b)\ \ D(a)=\{(x,y): 0\le x\le a, 0\le y\le a\}$

Hence prove that $(c)\ \ \int\limits_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$ $\qquad (d)\ \ \int\limits_0^\infty x^2e^{-x^2}dx=\frac{\sqrt{\pi}}{4}$

Solution: (a) Let $D_1(a)=\{(x,y): x^2+y^2\le a^2\}$. Then by using polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$, we have

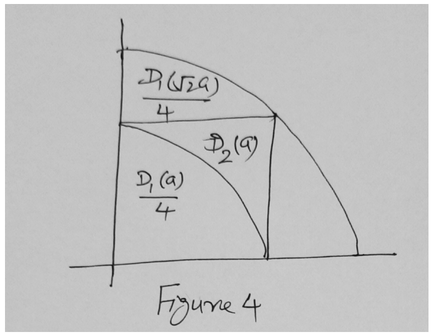
$$I_1(a)=\iint\limits_{D_1(a)} e^{-(x^2+y^2)}dxdy=\int_0^{2\pi}\int_{r=0}^{a} e^{-r^2}rdrd\theta=\pi(1-e^{-a^2})\to\pi.$$

(b) Write $D_2(a)=\{(x,y): 0\le x\le a, 0\le y\le a\}$. It is clear from Figure 4

that $\frac{D_1(a)}{4}<D_2(a)<\frac{D_1(\sqrt{2}a)}{4}$. Let $I_2(a)=\iint\limits_{D_2(a)} e^{-(x^2+y^2)}dxdy$. Then the corresponding integrals satisfy $\frac{I_1(a)}{4}<I_2(a)<\frac{I_1(\sqrt{2}a)}{4}$. By sandwich theorem, we get $\lim\limits_{a\to\infty} I_2(a)=\frac{\pi}{4}$.

(c) Let $I(a)=\int_0^a e^{-x^2}dx$. Then by Fubini's theorem,

$$I^2(a)=\left(\int_0^a e^{-x^2}dx\right)\left(\int_0^a e^{-y^2}dy\right)=\int_{x=0}^{a}\int_{y=0}^{a} e^{-(x^2+y^2)}dxdy=I_2(a)\to\frac{\pi}{4}.$$



(d) Let $J(a) = \int\limits_0^a x^2 e^{-x^2} dx$. Then by Fubini's theorem,

$$J^2(a) = \left(\int_0^a x^2 e^{-x^2} dx\right)\left(\int_0^a y^2 e^{-y^2} dy\right) = \int_{x=0}^a \int_{y=0}^a x^2 y^2 e^{-(x^2+y^2)} dxdy$$
$$= \iint_{D_2(a)} x^2 y^2 e^{-(x^2+y^2)} dxdy.$$

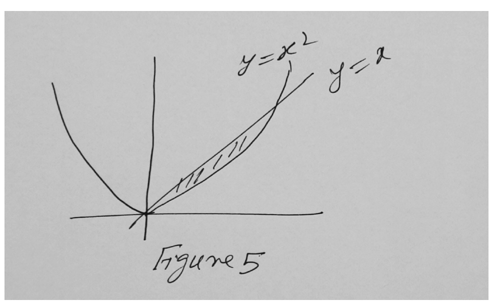
By using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, we can write

$$\iint_{D_1(a)} x^2 y^2 e^{-(x^2+y^2)} dxdy = \frac{1}{4}\int_0^{2\pi}\int_{r=0}^a r^4 (\sin 2\theta)^2 e^{-r^2} r dr d\theta.$$

Use similar argument as in solution of (b) to get answer in this case.

5. Let D denote the solid bounded by the surfaces $y = x,\ y = x^2,\ z = x$ and $z = 0$. Evaluate $\iiint\limits_D y dxdydz$.

Solution: Here $y = x,\ y = x^2,\ z = x$ and $z = 0$, implies $y = 0, 1$. Please see Figure 5.



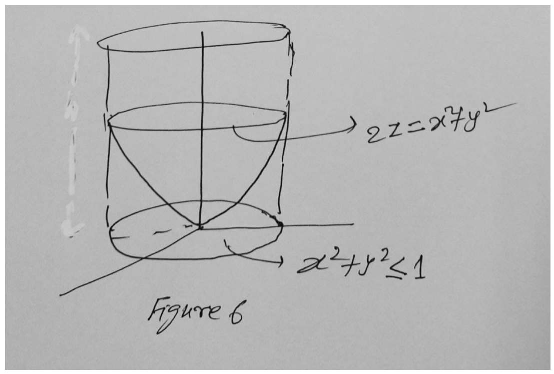
By Fubini's theorem, we get

$$\iiint\limits_D y dxdydz = \int_{x=0}^1 \left(\int_{z=0}^x \left(\int_{y=x^2}^x y dy\right) dz\right) dx.$$

6. Let D denote the solid bounded above by the plane $z = 4$ and below by the cone $z = \sqrt{x^2+y^2}$. Evaluate $\iiint\limits_D \sqrt{x^2+y^2+z^2} dxdydz$.

Solution: Use spherical polar coordinate $x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi$, where $0 \le \theta < 2\pi$ and $0 \le \phi < \frac{\pi}{4}$.

7. Find the surface integral $\iint\limits_S z d\sigma$, where S it the part of the paraboloid $2z = x^2 + y^2$ which lies in the cylinder $x^2 + y^2 = 1$.
Solution: Please see Figure 6.

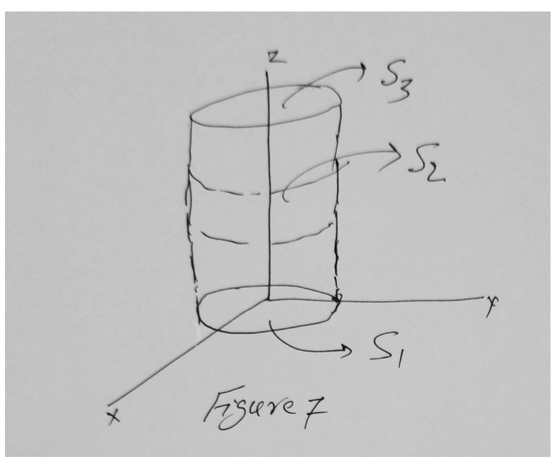


Let $z = f(x, y) = \frac{x^2+y^2}{2}$ and $D = x^2 + y^2 \le 1$.

$$\iint\limits_S z d\sigma = \iint\limits_D z\ \sqrt{1 + f_x^2 + f_y^2}\ dxdy.$$

Use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ to evaluate the integral on D.

8. What is the integral of the function $x^2 z$ taken over the entire surface of a right circular cylinder of height h which stands on the circle $x^2 + y^2 = a^2$.
Solution: We divide the surface of the cylinder into three parts $S_i; i = 1, 2, 3$ as shown in the Figure 7.



$$\iint\limits_S x^2 z \; d\sigma = \left(\iint\limits_{S_1} + \iint\limits_{S_2} + \iint\limits_{S_3} \right) x^2 z \; d\sigma.$$

Note that S_1 is the bottom of the cylinder given by $x^2 + y^2 \leq a^2$ and $z = 0$. Hence $\iint\limits_{S_1} x^2 z \; d\sigma = 0$. Here S_2 is the vertical surface given by $r(\alpha, \beta) = (a \cos \alpha, a \sin \alpha, \beta)$, where $0 \leq \alpha < 2\pi$ and $0 \leq \beta \leq h$. Hence

$$\iint\limits_{S_2} x^2 z \; d\sigma = \int_{\beta=0}^{h} \int_{\alpha=0}^{2\pi} (a \cos \alpha)^2 \beta \| r_\alpha \times r_\beta \| d\alpha d\beta = \frac{\pi a^3 h^2}{2}.$$

Here S_3 is the top of the cylinder given by $x^2 + y^2 \leq a^2$ and $z = h$. This can be parametrized by $r(u, v) = (u \cos v, u \sin v, h)$, where $0 \leq u \leq a$ and $0 \leq v < 2\pi$. Thus,

$$\iint\limits_{S_3} x^2 z \; d\sigma = \int_{u=0}^{a} \int_{v=0}^{2\pi} (u \cos v)^2 h \| r_u \times r_v \| du dv = \frac{\pi a^4 h}{4}.$$