Assignment 5

1. Let (X, S, μ) be a measure space and $0 < p < 1$. Then for $f, g \in L^+ \cap L^p(X, S, \mu)$ show that $\|f + g\|_p \geq \|f\|_p + \|g\|_p$.

2. Let (X, S, μ) be a measure space and $1 \leq p < \infty$. For $f \in L^p(X, S, \mu)$ and $\alpha > 0$ show that $\mu\{x \in X : |f(x)| \geq \alpha\} \leq \left(\frac{\|f\|_p}{\alpha}\right)^p$.

3. Let $1 \leq p < \infty$ $f \in L^p(\mathbb{R}, M, m)$. Then show that $\|f(x + h) - f(x)\|_p \to 0$ as $|h| \to 0$.

4. Let (X, S, μ) be a finite measure space and $1 \leq p < q \leq \infty$. For $f \in L^q(X, S, \mu)$, show that $\|f\|_p \leq (\mu(X))^{\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_q$. Further deduce that $L^q(X, S, \mu)$ is a proper dense subspace of $L^p(X, S, \mu)$.

5. Show that the space of all simple functions is dense in $L^\infty(X, S, \mu)$.

6. Suppose $f \in L^\infty(X, S, \mu)$ is supported on a set of finite measure. Then show that f is in $L^p(X, S, \mu)$ for all $p \geq 1$ and $\lim\limits_{p \to \infty} \|f\|_p = \|f\|_\infty$.

7. Prove that $L^1(\mathbb{R}, M, m) \cap L^p(\mathbb{R}, M, m)$ is a proper dense subspace of $L^p(\mathbb{R}, M, m)$, whenever $1 < p < \infty$.

8. Let $1 \leq p < q < r \leq \infty$ and $p^{-1} + q^{-1} = r^{-1}$. Show that for $f \in L^p(X, S, \mu)$ and $g \in L^q(X, S, \mu)$ $fg \in L^1(X, S, \mu)$ and $\|fg\|_r \leq \|f\|_p \|g\|_q$. (A generalized Holder's inequality.)

9. Let $1 \leq p < q < r \leq \infty$. Then $L^q(X, S, \mu) \subset L^p(X, S, \mu) + L^r(X, S, \mu)$.

10. Let $1 \leq p < q < r \leq \infty$. Show that $L^p(X, S, \mu) \cap L^r(X, S, \mu) \subset L^q(X, S, \mu)$ and $\|f\|^q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$, where $\lambda \in (0, 1)$ is given by $q^{-1} = \lambda p^{-1} + (1 - \lambda) r^{-1}$.

11. Let $1 \leq p < \infty$ and $p^{-1} + q^{-1} = 1$. For $f \in L^p(X, S, \mu)$, prove that
$$\|f\|_p = \sup \left\{ \left| \int_X f g d\mu \right| : g \in L^q(X, S, \mu) \text{ and } \|g\|_q = 1 \right\}.$$

12. Let $\mathcal{B}(\mathbb{R}^2)$ be the σ-algebra generated by Borel subsets of $\mathbb{R}^2$ (i.e. σ-algebra generated by open subsets of $\mathbb{R}^2$). Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

13. Let $f : (X, S, \mu) \to \mathbb{R}$ be measurable. Show that $G_f = \{(x, y) \in X \times \mathbb{R},\ y = f(x)\} \in S \otimes \mathcal{B}(\mathbb{R})$. If $(X, S, \mu) = (\mathbb{R}, M, m)$, then show that $m \times m(G_f) = 0$.

14. Let(X, S, μ) be a σ-finite measure space. Let $f : (X, S, \mu) \to [0, \infty]$ be measurable. Show that $A_f = \{(x, y) \in X \times [0, \infty],\ y \leq f(x)\} \in S \otimes \mathcal{B}(\mathbb{R})$ and $\mu \times m(A_f) = \int_X f(x) d\mu(x)$.

15. Let $f(x, y) = e^{-xy} \sin x$ and $D = [0, \infty) \times [1, \infty)$. Show that $f\chi_D \in L^1(\mathbb{R}^2, M \otimes M, m \times m)$ and
$$\int_0^\infty \int_1^\infty f(x, y) dy dx = \int_1^\infty \int_0^\infty f(x, y) dx dy.$$

16. Let $f(x, y) = e^{-xy} - 2e^{-2xy}$ and $D = [0, 1] \times [1, \infty)$. Show that $f\chi_D \notin L^1(\mathbb{R}^2, M \otimes M, m \times m)$.

17. Let $f \in L^1(X, S, \mu)$ and $g \in L^1(Y, T, \nu)$. Define $\varphi(x, y) = f(x)g(y)$. Show that φ is measurable and $\varphi \in L^1(X \times Y, S \otimes T, \mu \times \nu)$.

18. Let $f \in L^1(0,a)$ and define $g(x) = \int_x^a \frac{f(t)}{t} dm(t)$. Then show that $g \in L^1(0,a)$ and compute $\int_0^a g(x)dm(x)$.

19. Let $X = Y = [0,1]$, $S = T = \mathcal{B}[0,1]$ and $\mu = \nu = m$. Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 2y & \text{if } y \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Compute $\int_0^1 \int_0^1 f(x,y)dydx$ and $\int_0^1 \int_0^1 f(x,y)dxdy$. Whether $f \in L^1(m \times m)$?