MA15010H: Multi-variable Calculus

(Assignment 5 Hint/model solutions: Riemann Integration, Fubini's Theorem)
September - November, 2025

1. If $f : D = [a,b] \times [c,d] \to \mathbb{R}$ is continuous, then f is uniformly continuous.
Solution: Suppose f is not uniformly continuous on D. Then, there is an $\epsilon > 0$ such that for each $\delta = \frac{1}{n}, n \in \mathbb{N}$, there exist sequences X_n, and Y_n in D such that $\|X_n - Y_n\| < \frac{1}{n}$ but $|f(X_n) - f(Y_n)| \geq \epsilon$. Since D is closed and bounded, by Bolzano-Weierstrass Theorem, there will be subsequence X_{n_k} such that $X_{n_k} \to X \in D$. Similarly, Y_{n_k} has subsequence $Y_{n_{kl}}$ such that $Y_{n_{kl}} \to Y \in D$. Hence, without lose of generality, we can assume that $X_{n_k} \to X$ and $Y_{n_k} \to Y$. Thus, we have $\|X_{n_k} - Y_{n_k}\| < \frac{1}{n_k}$ and $|f(X_{n_k}) - f(Y_{n_k})| \geq \epsilon$. It follows that $X = Y$. By continuity of f at X and Y, we get $|f(X) - f(Y)| \geq \epsilon$, which is a contradiction.
2. Let f be real valued continuous function on $[a,b]$. Show that the graph of f is a set of content zero.
Solution: Let $G_f = \{(x, f(x)) : \ x \in [a,b]\}$. Note that the function f is uniformly continuous on $[a,b]$. For given $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies

$$|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}. \tag{0.1}$$

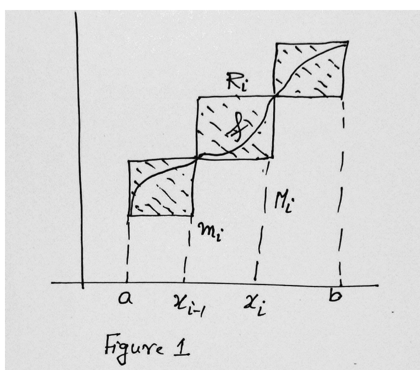


Figure 1

Let $P = \{x_0, \ldots, x_{i-1}, x_i, \ldots, x_n\}$ be a partition of $[a,b]$ such that $\Delta x_i < \delta$. Then (0.1) will be satisfied by every pair of points $x, y \in [x_{i-1}, x_i]$. That is,

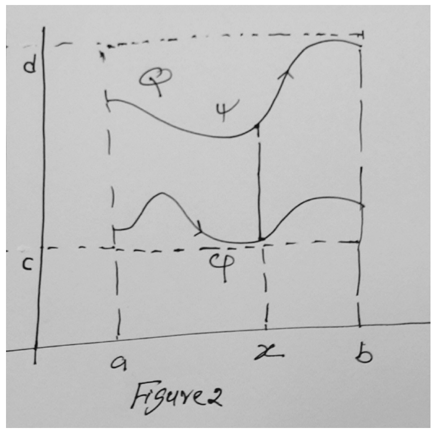
$$-\frac{\epsilon}{2(b-a)} < f(x) - f(y) < \frac{\epsilon}{2(b-a)}.$$

By taking supremum w.r.t. $x \in [x_{i-1}, x_i]$ keeping y fixed and then supremum w.r.t. y, we get $M_i - m_i < \frac{\epsilon}{2(b-a)}$. Note that $(M_i - m_i)\Delta x_i$ is the area of the rectangle $R_i = [m_i, M_i] \times [x_{i-1}, x_i]$ along the graph of f as shown in Figure 1. This shows that $\sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \epsilon$. Thus, $G_f \subset \bigcup_{i=1}^{n} R_i$ and Area$(\bigcup_{i=1}^{n} R_i) < \epsilon$. Hence G_f is of content zero.

3. Let $D = \{(x,y) : a \le x \le b \text{ and } \varphi(x) \le y \le \psi(x)\}$, where φ and ψ are continuous functions on $[a,b]$. If f is a bounded continuous functions on D, then

$$\iint\limits_D f(x,y)dxdy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy \right) dx.$$

Solution: Since φ and ψ are continuous on $[a,b]$, they are bounded and hence D is a bounded domain in $\mathbb{R}^2$.



Let $Q = [a,b] \times [c,d]$ be a rectangle containing D as shown in Figure 2. Extend f on Q as $\tilde{f} : Q \to \mathbb{R}$, where

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in Q \setminus D. \end{cases}$$

By definition of $\tilde{f}$, it is clear that $\tilde{f}$ is continuous on the interior of D. It is clear from Figure 2, the domain D is bounded by the graph of φ, ψ and two vertical line segments, each of content zero. Hence $\tilde{f}$ has discontinuities in Q of content zero. Thus, $\tilde{f}$ is integrable. Now, it only remain to show that

$$\iint\limits_Q \tilde{f}(x,y)dxy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy \right) dx.$$

Note that for each fixed $x \in [a,b]$, the integral $\int\limits_c^d \tilde{f}(x,y)dy$ exists, since the set of discontinuities of $\tilde{f}(x,\cdot)$ contains at most two points, one each on the graph of φ and ψ. Moreover, $G(x) = \int\limits_c^d \tilde{f}(x,y)dy$ is continuous expect possibly at a and b. Hence G is integrable on $[a, b]$. By applying Fubini's Theorem to $\tilde{f}$ on Q, we get

$$\iint\limits_D f(x,y)dxdy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} \tilde{f}(x,y)dy \right) dx.$$

But, this follows from the fact that

$$\int_c^d \tilde{f}(x,y)dy = \int_{\varphi(x)}^{\psi(x)} f(x,y)dy.$$

Hence the result followed.

4. Evaluate the following integral applying Fubini's Theorem

(a) $\int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} \sqrt{1-y^2}dydx$

(b) $\int\limits_0^\pi \int\limits_x^\pi \frac{\sin y}{y} dydx$

(c) $\int\limits_0^1 \int\limits_y^1 x^2 e^{xy} dxdy$

Solution: (a) The domain of integration is as shown Figure 3.

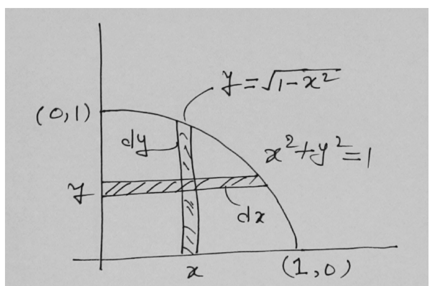


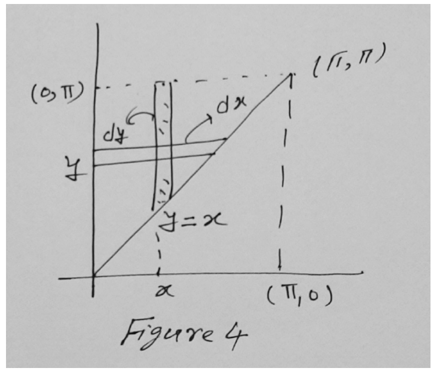
Figure 3

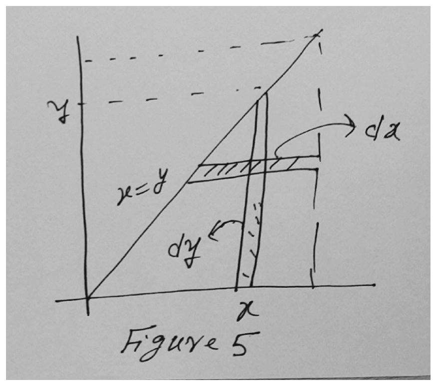
$$\int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} \sqrt{1-y^2}dydx = \int\limits_{y=0}^1 \left(\int\limits_{x=0}^{\sqrt{1-y^2}} \sqrt{1-y^2}dx \right) dy = \int\limits_{y=0}^1 (1-y^2)dy.$$

(b) The domain of integration is as shown Figure 4.

$$\int\limits_0^\pi \left(\int\limits_x^\pi \frac{\sin y}{y} dy \right) dx = \int\limits_{y=0}^\pi \left(\int\limits_{x=0}^y \frac{\sin y}{y} dx \right) dy = \int\limits_{y=0}^\pi \sin y \; dy.$$

(c) The domain of integration is as shown Figure 5.

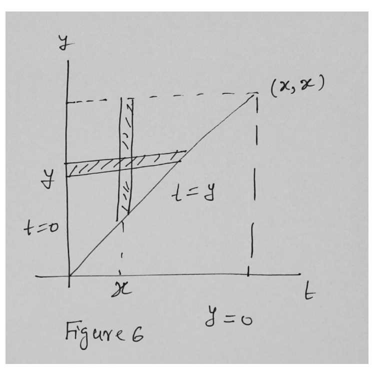




$$\int_0^1 \int_y^1 x^2 e^{xy} dx dy = \int_{x=0}^{1} \left(\int_{y=0}^{x} x^2 e^{xy} dy \right) dx.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that $\int_{y=0}^{x} \int_{t=0}^{y} f(t) dt dy = \int_{t=0}^{x} (x - t) f(t) dt$.

Solution: The domain of integration is as shown Figure 6.



$$\int_{y=0}^{x} \left(\int_{t=0}^{y} f(t) dt \right) dy = \int_{t=0}^{x} \left(\int_{y=t}^{x} f(t) dy \right) dt = \int_{t=0}^{x} (x - t) f(t) dt.$$

6. Let f be a continuous function on the bounded domain D. If $\iint\limits_R f(x,y)dxdy = 0$ for all rectangle R in D, then $f = 0$ on D.
Solution: Suppose there exists $X_o \in D$ such that $f(X_o) \neq 0$. Then without loss of generality we can assume that $f(X_o) > 0$. Since f is continuous at X_o, for $\epsilon = \frac{f(X_o)}{2} > 0$, there exists an open ball $B_\delta(X_o)$ such that $|f(X) - f(X_o)| < \frac{f(X_o)}{2}$. This implies $f(X) > \frac{3f(X_o)}{2}$ for each $X \in B_\delta(X_o)$. Thus,
$$\iint\limits_R f(x,y)dxdy = 0$$
for each rectangle $R \subset B_\delta(X_o)$. Since f is continuous on R, it follows that f must be zero on R. If not, then suppose, $f(y_o) > 0$ for some $Y_o \in R$. Then there exists a ball $B_r(Y_o) \subset R$ such that $f(X) > \frac{3f(Y_o)}{2}$ for each $X \in B_r(Y_o)$. But, then
$$0 = \iint\limits_R f(x,y)dxdy > \iint\limits_{B_r(Y_o)} f(x,y)dxdy \geq \frac{3f(Y_o)}{2} \iint\limits_{B_r(Y_o)} dxdy = \frac{3f(Y_o)}{2}\pi r^2 > 0.$$
which is a contradiction.
7. Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be a continuous function. If f_x, f_y, f_{xy} and f_{yx} are continuous then, by using Fubini's theorem, show that $f_{xy} = f_{yx}$.
Solution: Since f_{xy} is continuous on D, by Fubini's Theorem, we get
$$\begin{aligned}
\int_a^x \int_c^y \frac{\partial^2 f}{\partial x \partial y}(u,v)dvdu &= \int_c^y \int_a^x \frac{\partial^2 f}{\partial x \partial y}(u,v)dudv \\
&= \int_c^y \left[\frac{\partial f}{\partial y}(x,v) - \frac{\partial f}{\partial y}(a,v)\right] dv \\
&= f(x,y) - f(x,c) - f(a,y) + f(a,c).
\end{aligned}$$
Also,
$$\int_a^x \int_c^y \frac{\partial^2 f}{\partial y \partial x}(u,v)dvdu = f(x,y) - f(x,c) - f(a,y) + f(a,c).$$
Hence
$$\int_a^x \int_c^y \frac{\partial^2 f}{\partial x \partial y}(u,v)dvdu = \int_a^x \int_c^y \frac{\partial^2 f}{\partial y \partial x}(u,v)dvdu.$$
Since the above equation holds for every choice of $x, y \in D$, we obtain
$$\iint\limits_R \frac{\partial^2 f}{\partial x \partial y}(u,v)dvdu = \iint\limits_R \frac{\partial^2 f}{\partial y \partial x}(u,v)dvdu$$
for every rectangle $R \subseteq D$. Thus, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.