Assignment 4

1. Let $1 \leq p < \infty$. Define a linear map $T : l^p \to l^p$ by $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Find the adjoint operator T^* of T.

2. Show that the linear map $T : (C^1[0,1], \|.\|) \to (C[0,1], \|.\|)$ defined by $(Tf)(t) = f'(t)$ does not have continuous adjoint.

3. Let X and Y be two normed linear spaces. Suppose $T \in B(X,Y)$. Show that $T^* \in B(Y^*, X^*)$ and $\|T^*\| = \|T\|$.

4. Let $(X, \langle .,. \rangle)$ be an inner product space. Prove the following generalized parallelogram law.
$$\sum_{\epsilon_k = \pm 1} \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\|^2 = 2^n \sum_{k=1}^{n} \|x_k\|^2 .$$

5. Let ω be a nth root of unity. Then Show that for x, y in an inner product space X following holds.
$$\langle x, y \rangle = \frac{1}{n} \sum \omega^p \|x + \omega^p y\|^2 .$$

6. Let X be an inner product space, let $x \in X$ and let (x_n) be a sequence in X such that $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle$. Show that $x_n \to x$ in X.

7. Let X be an inner product space and (x_n) and (y_n) be sequences in $\overline{B(0,1)}$ such that
$\|\frac{1}{2}(x_n + y_n)\| \to 1$ as $n \to \infty$. Show that $\|x_n - y_n\| \to 0$ as $n \to \infty$.

8. Let X be an inner product space and let $y, z \in X$. If $Tx = \langle x, y \rangle z$ for all $x \in X$, then show that $\|T\| = \|y\|\|z\|$.

9. Consider $C_{\mathbb{R}}[0,1]$ with the usual inner product. Let $S = \{p_n : n = 0, 1, 2, ...\}$, where $p_n(t) = t^n$ for all $t \in [0,1]$ and for $n = 0, 1, 2, ...$ Prove that the orthogonal complement of S in $C_{\mathbb{R}}[0,1]$ is $\{0\}$.

10. Let M be a closed subspace of a Hilbert space H. If $x \in M$ and if (x_n) is a sequence in M, then show that $x_n \xrightarrow{w} x$ in H iff $x_n \xrightarrow{w} x$ in M.

11. Using Riesz representation theorem, show that $\{(x_n) \in \ell^2 : \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x_n = 0\}$ is not a closed subset of the Hilbert space ℓ^2.

12. Determine $\|f\|$ for the linear functional $f : (\ell^2, \|\cdot\|_2) \to \mathbb{K}$, defined by $f((x_n)) = \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{n(n+1)}}$ for all $(x_n) \in \ell^2$.

13. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space H. If $f(x) = \sum_{n=1}^{\infty} \frac{1}{3^n} \langle x, e_n \rangle$ for all $x \in H$, then determine $\|f\|$.

14. Let H be a Hilbert space and let (T_n) be a sequence in $B(H)$ such that for each $x, y \in H$, $\lim_{n\to\infty} \langle T_n x, y \rangle$ exists in $\mathbb{K}$. Show that $\sup\{\|T_n\| : n \in \mathbb{N}\} < \infty$.

15. Let $(X, \|\cdot\|)$ be a separable Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. If $\|x\|_0 = \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle x, e_n \rangle|$ for all $x \in X$, then show that $\|\cdot\|_0$ is a norm on X which is not equivalent to $\|\cdot\|$.

16. Let $T : L^2[0,1] \to L^2[0,1]$ be a linear map which is defined by

$$(Tf)(x) = \int_0^x f(t)dt.$$

Define $\langle Tf, g \rangle = \langle f, T^* \rangle$. Find the adjoint operator T^* of T.

17. Let T be linear operator on a Hilbert space H such that $(Tx, y) = (x, Ty)$. Show that T is continuous.

18. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for a separable Hilbert space H. Define a linear map $T : H \to H$ by $T(e_n) = a_n e_n$, $n = 1, 2, \ldots$. Show that T is bounded if and only if sequence $\{a_n\}$ is bounded.

19. Let T be a normal operator on a Hilbert space H. Show that $\|T^2\| = \|T\|^2$.