MA642: Real Analysis -1

(Assignment 4: Functions of several variables)
January - April, 2025

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) There exists a one-one continuous function from $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ onto $\mathbb{R}^2$.
   (b) There exists a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is differentiable only at $(1,0)$.
   (c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f_x(0,0) = 0$. Then there exists some $\delta > 0$ such that $f(x,0)$ is continuous on $(-\delta, \delta)$.
   (d) If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with $f(0,0) = (1,1)$ and $[f'(0,0)] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then there cannot exist a differentiable function $g : \mathbb{R}^2 \to \mathbb{R}^2$ with $g(1,1) = (0,0)$ and $(f \circ g)(x,y) = (y,x)$ for all $(x,y) \in \mathbb{R}^2$.
   (e) A continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ cannot be one-one and onto if $\det[f'(x,y)] = 0$ for some $(x,y) \in \mathbb{R}^2$.
   (f) The equation $\sin(xyz) = z$ defines x implicitly as a differentiable function of y and z locally around the point $(x,y,z) = (\frac{\pi}{2}, 1, 1)$.

2. Let Ω be an open subset of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^m$ and $g : \Omega \to \mathbb{R}^m$ be continuous at $\mathbf{x}_0 \in \Omega$. If for each $\varepsilon > 0$, there exist $\mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}_0)$ such that $f(\mathbf{x}) = g(\mathbf{y})$, then show that $f(\mathbf{x}_0) = g(\mathbf{x}_0)$.

3. Let $A(\neq \emptyset) \subset \mathbb{R}^n$ be such that every continuous function $f : A \to \mathbb{R}$ is bounded. Show that A is a closed and bounded subset of $\mathbb{R}^n$.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and let $f(\mathbf{x}) = T(\mathbf{x}) \cdot \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Find $f'(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$.

5. Examine the differentiability of f at $\mathbf{0}$, where
   (a) $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $|f(\mathbf{x})| \le \|\mathbf{x}\|_2^2$ for all $\mathbf{x} \in \mathbb{R}^n$.
   (b) $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$.
   (c) $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|_2 \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

6. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in \Omega$, let $f(\mathbf{x}_0) = 0$ and let $g : \Omega \to \mathbb{R}$ be continuous at $\mathbf{x}_0$. Prove that $fg : \Omega \to \mathbb{R}$, defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x} \in \Omega$, is differentiable at $\mathbf{x}_0$.

7. Let Ω be a nonempty open subset of $\mathbb{R}^n$ and let $g : \Omega \to \mathbb{R}^n$ be continuous at $\mathbf{x}_0 \in \Omega$. If $f : \Omega \to \mathbb{R}$ is such that $f(\mathbf{x}) - f(\mathbf{x}_0) = g(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0)$ for all $\mathbf{x} \in \Omega$, then show that f is differentiable at $\mathbf{x}_0$.

8. The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$ in the directions of $(1,2)$ and $(2,1)$ are 1 and 2 respectively. Find $f_x(0,0)$ and $f_y(0,0)$.

9. Let $A \in GL(\mathbb{R}^n)$ and $\alpha \ge 2$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\|f(x)\| \le k\|x\|^\alpha$, for some $k > 0$. Prove/disprove that the map $g = f + A$ is continuously differentiable at $\mathbf{0}$ and g is invertible in the neighborhood of $\mathbf{0}$.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable such that $f(1,1) = 1$, $f_x(1,1) = 2$ and $f_y(1,1) = 5$. If $g(x) = f(x, f(x,x))$ for all $x \in \mathbb{R}$, determine $g'(1)$.

11. Prove that a differentiable function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^m$ is homogeneous of degree $\alpha \in \mathbb{R}$ (*i.e.* $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and for all $t > 0$) iff $f'(\mathbf{x})(\mathbf{x}) = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

12. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is satisfying $f(rx) = r^{\frac{3}{2}} f(x)$ for all $(x, r) \in \mathbb{R}^n \times (0, \infty)$. Whether f is differentiable at $\mathbf{0}$?

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that $f_x(a, b) = f_y(a, b)$ for all $(a, b) \in \mathbb{R}^2$ and $f(a, 0) > 0$ for all $a \in \mathbb{R}$. Show that $f(a, b) > 0$ for all $(a, b) \in \mathbb{R}^2$.

14. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be such that $f(tx) = t^2 f(x)$ for every $t > 0$ and $x \in \mathbb{R}^n$. Does it imply that f is differentiable at 0?

15. Let Ω be an open subset of $\mathbb{R}^n$ such that $\mathbf{a}, \mathbf{b} \in \Omega$ and $S = \{(1-t)\mathbf{a} + t\mathbf{b} : t \in [0, 1]\} \subset \Omega$. If $f : \Omega \to \mathbb{R}^m$ is differentiable at each point of S, then show that there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(\mathbf{b}) - f(\mathbf{a}) = L(\mathbf{b} - \mathbf{a})$.

16. Let $f(x, y) = (2ye^{2x}, xe^y)$ for all $(x, y) \in \mathbb{R}^2$. Show that there exist open sets U and V in $\mathbb{R}^2$ containing $(0, 1)$ and $(2, 0)$ respectively such that $f : U \to V$ is one-one and onto.

17. Let $f(x, y) = (3x - y^2, 2x + y, xy + y^3)$ and $g(x, y) = (2ye^{2x}, xe^y)$ for all $(x, y) \in \mathbb{R}^2$. Examine whether $(f \circ g^{-1})'(2, 0)$ exists (with a meaningful interpretation of g^{-1}) and find $(f \circ g^{-1})'(2, 0)$ if it exists.

18. For $n \geq 2$, let $B = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < 1\}$ and let $f(\mathbf{x}) = \|\mathbf{x}\|_2^2 \mathbf{x}$ for all $\mathbf{x} \in B$. Show that $f : B \to B$ is differentiable and invertible but that $f^{-1} : B \to B$ is not differentiable at $\mathbf{0}$.

19. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a differentiable function with $\|f'(x)\| \leq 1$. Show that f satisfies $\|f(x) - f(y)\| \leq |x - y|$ for every $x, y \in \mathbb{R}$. (Hint: use one dimensional MVT.)

20. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Find an appropriate condition such that $f(x, (f(x, y)) = 0$ can be solved for x in some neighborhood of $(0, 0)$.

21. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $f'(0) \neq 0$. Show that $F(x, y) = (x - yf(y), f(y))$ is locally invertible in some neighborhood of $(0, 0)$. Does there exists some f for which F is globally invertible?

22. Using implicit function theorem, show that the system of equations

$$x^3(y^3 + z^3) = 0,$$
$$(x - y)^3 - z^2 = 7,$$

can be solved locally near the point $(1, -1, 1)$ for y and z as a differentiable function of x.

23. Using implicit function theorem, show that in a neighbourhood of any point $(x_0, y_0, u_0, v_0) \in \mathbb{R}^4$ which satisfies the equations

$$x - e^u \cos v = 0,$$
$$v - e^y \sin x = 0,$$

there exists a unique solution $(u, v) = \varphi(x, y)$ satisfying $\det[\varphi'(x, y)] = v/x$.

24. Show that around the point $(0, 1, 1)$, the equation $xy - z \log y + e^{xz} = 1$ can be solved locally as $y = f(x, z)$ but cannot be solved locally as $z = g(x, y)$.

25. Find the 3rd order Taylor polynomial of $f(x, y, z) = x^2y + z$ about the point $(1, 2, 1)$.

26. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Show that f is not one-one.