

## Assignment 4

1. Let  $f \in L^1(X, S, \mu)$  and  $E_n = \{x \in X : |f(x)| \geq n\}$ . Show that  $\lim_{n \rightarrow \infty} n \cdot \mu(E_n) = 0$ .
2. Let  $f \in L^1(X, S, \mu)$ , then show that for each  $\epsilon > 0$  there exist  $\delta > 0$  and set  $E \in S$  such that  $\int_E |f| d\mu < \epsilon$ , whenever  $\mu(E) < \delta$ .
3. Let  $\mu(X) < \infty$  and  $f_n \in L^1(X, S, \mu)$  such that  $f_n \rightarrow f$  uniformly. Show that  $f \in L^1(X, S, \mu)$  and  $\int_X f = \lim \int_X f_n$ .
4. Let  $\mu(X) < \infty$  and  $f : X \rightarrow [0, \infty]$  be a measurable function. Show that  $f \in L^1(X, S, \mu)$  if and only if  $\sum_{n=0}^{\infty} \mu\{x \in X : f(x) \geq n\} < \infty$ .
5. Let  $f_n : X \rightarrow [0, \infty]$  be a decreasing sequence of measurable functions and  $f_n \rightarrow f$  point wise. If  $f_1 \in L^1(X, S, \mu)$ . Then show that  $\int_X f = \lim \int_X f_n$ .
6. Let  $f \in L^1(\mathbb{R}, M, m)$  be such that  $\int_I f = 0$ , for any open interval  $I \subset \mathbb{R}$ , then show that  $f = 0$  a.e. on  $X$ .
7. Let  $f_n, g : X \rightarrow \overline{\mathbb{R}}$  be measurable functions such that  $f_n \leq g$ ,  $\forall n \in \mathbb{N}$  and  $g \in L^1(X, S, \mu)$ . Show that  $\limsup \int_X f_n \leq \int_X \limsup f_n$ .
8. Let  $f_n : X \rightarrow [0, \infty]$  be a sequence of measurable functions and  $f_n \rightarrow f$  point wise such that  $\int_X f = \lim \int_X f_n < \infty$ . Show that  $\int_E f = \lim \int_E f_n$ , for any  $E \in S$ .
9. Let  $f_n, f, g_n, g \in L^1(X, S, \mu)$  be such that  $|f_n| \leq g_n$ ,  $f_n \rightarrow f$  and  $g_n \rightarrow g$  point wise. Show that  $\int_X g = \lim \int_X g_n$  implies  $\int_X f = \lim \int_X f_n$ .
10. Let  $f_n, f \in L^1(X, S, \mu)$  be such that  $f_n \rightarrow f$  point wise. Prove that  $\lim \int_X |f_n - f| = 0$  if and only if  $\int_X |f| = \lim \int_X |f_n|$ . (Hint: Use question 9.)
11. Let  $\mu$  be the counting measure on the measurable space  $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$  and let  $f : \mathbb{N} \rightarrow [0, +\infty]$ . Show that  $\int_E f d\mu = \sum_{n \in E} f(n)$  for every  $E \subset \mathbb{N}$  and hence, in particular,  $\int_{\mathbb{N}} f d\mu = \sum_{n=1}^{\infty} f(n)$ .
12. Let  $X$  be a set,  $x \in X$  and let  $\delta_x$  be the Dirac measure at  $x$  on the measurable space  $(X, \mathcal{P}(X))$ . If  $f : X \rightarrow [0, +\infty]$  and  $E \subset X$ , then show that  $\int_E f d\delta_x = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$   
(Hence, in particular,  $\int_X f d\delta_x = f(x)$ .)
13. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f = \frac{1}{\sqrt{x}} \chi_{(0,1)}$ . Let  $g(x) = \sum_{r_n \in \mathbb{Q}} 2^{-n} f(x - r_n)$ , then show that the function  $g$  belongs to  $L^1(\mathbb{R}, M, m)$ .
14. For each  $x \in [0, 1]$ , let  $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{k}{n} \text{ for some } k, n \in \mathbb{N} \text{ with g.c.d.}(k, n) = 1, \\ 0 & \text{otherwise.} \end{cases}$   
Evaluate the Lebesgue integral  $\int_{[0,1]} f dm$ .
15. Let  $(X, \mathcal{S}, \mu)$  be a measure space and let  $f, g : X \rightarrow [0, +\infty]$  be  $\mathcal{S}$ -measurable. If  $\lambda(E) = \int_E f d\mu$  for all  $E \in \mathcal{S}$ , then show that  $\lambda$  is a measure on  $(X, \mathcal{S})$  and that  $\int_X g d\lambda = \int_X gf d\mu$ .

16. For each  $x \in [0, 1]$ , let  $f(x) = \begin{cases} x^2 & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ x^3 & \text{if } x = \frac{1}{3^n} \text{ for some } n \in \mathbb{N}, \\ x^4 & \text{otherwise.} \end{cases}$

Evaluate the Lebesgue integral  $\int_{[0,1]} f \, dm$ .

17. Let  $f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in [0, \frac{1}{2}] \setminus C, \\ \cos(\pi x) & \text{if } x \in (\frac{1}{2}, 1] \setminus C, \\ x^2 & \text{if } x \in C. \end{cases}$

Evaluate the Lebesgue integral  $\int_{[0,1]} f \, dm$ .

(Here  $C$  denotes the Cantor set.)

18. Evaluate the Lebesgue integrals: (a)  $\int_{[0,+\infty)} e^{-[x]} \, dx$       (b)  $\int_{(0,1]} \frac{1}{\sqrt[3]{x}} \, dx$

19. Let  $f(x) = \begin{cases} e^{[x]} & \text{if } x \in \mathbb{Q}, \\ e^{-[x]} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Evaluate the Lebesgue integral  $\int_{(0,+\infty)} f \, dm$ .

20. Let  $f(x) = \begin{cases} e^{|x|} & \text{if } x \in \mathbb{Q}, \\ e^{-|x|} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Evaluate the Lebesgue integral  $\int_{\mathbb{R}} f \, dm$ .

21. Let  $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1, \\ \frac{1}{x} & \text{if } x > 1. \end{cases}$

Evaluate the Lebesgue integral  $\int_{(0,+\infty)} f \, dm$ .

22. Evaluate the following: (a)  $\lim_{n \rightarrow \infty} \int_{-2}^2 \frac{x^{2n}}{1+x^{2n}} \, dx$       (b)  $\lim_{n \rightarrow \infty} \int_{[0,1]} \frac{1+nx}{(1+x)^n} \, dx$       (c)  $\int_0^1 \left( \sum_{n=1}^{\infty} \frac{x^n}{n} \right) \, dx$   
(d)  $\lim_{n \rightarrow \infty} \int_1^{\infty} \frac{1}{1+x^{2n}} \, dx$       (e)  $\sum_{n=0}^{\infty} \int_0^1 \frac{x^2}{(1+x^2)^n} \, dx$