Assignment 3

1. Let $C_c(\mathbb{R})$ be the class of all compactly supported continuous functions on $\mathbb{R}$. Does the linear functional given by $T(f) = \int\limits_{-\infty}^{\infty} f(t)dt$ is continuous?

2. Let f be a linear functional on a normed linear space X. Then f is bounded if and only if $\ker f$ is closed.

3. Let X^* be the dual space of a normed linear space X. For $x \in X$, show that $\|x\| = \sup\{|f(x)| : f \in X^* \text{ and } \|f\| = 1\}$.

4. Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L^p(\mathbb{R})$, prove that
$$\|f\|_p = \sup\left\{\left|\int_{\mathbb{R}} f(x)g(x)dx\right| : g \in L^q(\mathbb{R}) \text{ and } \|g\|_q = 1\right\}.$$

5. Define a family of linear functionals $f_n : c_o(\mathbb{N}) \to \mathbb{C}$ by $f_n(x) = \frac{1}{n}\sum_{j=1}^{n} x_j$. Show that $\lim\limits_{n\to\infty} f_n(x) = 0$ but $\|f_n\| = 1$.

6. Let $\{e_1, e_2 \ldots, e_n\}$ be a linearly independent set in an infinite dimentional normed linear space X. For $(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$, prove that there exists $f \in X^*$ such that $f(e_j) = a_j$, for $j = 1, 2, \ldots, n$.

7. Let $g_n \in L^2[0,1]$ be defined by
$$g_n(t) = \begin{cases} \sqrt{n} & \text{if } 0 \leq t < 1/n, \\ 0 & \text{if } 1/n \leq t \leq 1. \end{cases}$$
Show that $\|g_n\|_2 = 1$ and g_n converges weakly to 0.

8. For a 2π-periodic function $f \in L^2[-\pi, \pi]$, define a sequence (φ_n) of linear functionals by
$$\varphi_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt.$$
Show that $\|\varphi_n\| = 1$ and $\varphi_n(f) \to 0$.

9. Let $X = c_o$. Show that $X^* = l^1$ and $X^{**} = l^\infty$. For $x = (x_n) \in X$, prove that $x \mapsto \sum_1^\infty x_n$ is weakly continuous but not weak* continuous.

10. Let M be a proper subspace of a normed linear subspace X. Suppose $x_o \notin M$ and $\inf_{m\in M}\|x_o - m\| = \delta > 0$. Then there exist $f \in X^*$ such that $\|f\| = 1$, $f(x_o) = \delta$ and $f(x) = 0$, $\forall x \in M$. Does such f exist uniquely?

11. Let X and Y be two Banach spaces and $T_n, T \in B(X,Y)$ such that $T_n \to T$ weakly. Then $\sup\limits_n \|T_n\| < \infty$.

12. Let $X = (C[0,1], \|\cdot\|_\infty)$. Define a map $T : X \to \mathbb{C}$ by $T(f) = \int\limits_0^1 tf(t)dt$, for all $f \in X$. Find a vector $f_o \in X$ such that $T(f_o) = \|T\|$.

13. Let X and Y be two Banach spaces and $T : X \to Y$ such that $f \circ T \in X^*$ for each $f \in Y^*$. Then T is continuous.

14. Let X and Y be two normed linear spaces and $T \in B(X,Y)$. Define the linear map $T^* : Y^* \to X^*$ by $T^*(f) = f \circ T$, for all $f \in Y^*$. Show that

    (a) $\ker T^* = (\mathrm{Im} T)^\perp$.

    (b) T is bijective then T^* is bijective.

15. Let X and Y be two Banach spaces and $S : X \to Y$ and $T : Y^* \to X^*$ are linear maps such that $f \circ S = T(f)$, for all $f \in Y^*$. Then show that S continuous. (Hint: use close graph theorem).

16. Let X and Y be two Banach spaces and $T \in B(X,Y)$ such that range $\mathcal{R}(T)$ is closed. Prove that $\mathcal{R}(T^*) = (\ker T)^\perp$, where $M^\perp = \{f \in X^* : \; f(x) = 0, \; \forall x \in M\}$, for $M \subseteq X$.

17. Let X be a reflexive Banach space and $f \in X^*$. Show that there exists $x_o \in \overline{B(0,1)}$ such that $f(x_o) = \|f\|$.

18. Let K be a closed bounded convex subset of a reflexive Banach space X. Prove that K is weakly compact.

19. Suppose M is a subspace of a Banach space X. Then $M^\perp$ is weak* closed subspace of X^*.