MA746: Fourier Analysis

(Assignment 3: Distribution Theory)

July – November, 2025

1. (a) If Λ' is a compactly supported distribution, must it follow that Λ itself is compactly supported?
   (b) Is every compactly supported distribution necessarily of finite order?
   (c) Must the Fourier transform of every compactly supported function in $L^1(\mathbb{R})$ be real analytic?
   (d) Determine the distributional support of the function $\chi_{\mathbb{Q}}$, where $\mathbb{Q}$ denotes the set of rational numbers.
   (e) For $n \in \mathbb{N}$, let δ_n denote the Dirac delta distribution at n. Does $\delta_n \to 0$ in the weak* topology of $C_0(\mathbb{R})$ (the space of continuous functions vanishing at infinity)?
   (f) Determine the order of $\Lambda \in \mathcal{D}'(\mathbb{R})$ defined by
$$\Lambda(\varphi) = \int_{|x|>1} \log(x)\, \varphi(x)\, dx.$$
2. Suppose f is a continuous function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} f\varphi = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Show that $f = 0$.
3. Let $\Lambda = \Lambda_f$, where f is a continuous function on $\mathbb{R}^n$. Show that $\operatorname{supp} \Lambda_f = \operatorname{supp} f$. Does the same statement remain valid for locally integrable functions?
4. Show that there exists $\psi \in \mathcal{D}(\mathbb{R})$ such that $\varphi = \psi^{(k)}$ if and only if
$$\int_{\mathbb{R}} p(x)\varphi(x)\, dx = 0$$
for each polynomial p of degree at most $k-1$.
5. If $\Lambda \in \mathcal{D}'(\mathbb{R})$ satisfies $\Lambda' = 0$, prove that $\Lambda = \Lambda_c$ for some constant c.
6. Show that every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ can be written as
$$\varphi = \psi' + c\varphi_0,$$
where φ_0 is a fixed test function in $\mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi_0 \neq 0$.
7. Show that every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ can be written as
$$\varphi = x\psi + c\varphi_0,$$
where φ_0 is a fixed test function in $\mathcal{D}(\mathbb{R})$ with $\varphi_0(0) \neq 0$. Deduce that if $\Lambda \in \mathcal{D}'(\mathbb{R})$ and $x\Lambda = 0$, then $\Lambda = c\delta_0$.
8. Determine all $f \in C^\infty(\mathbb{R})$ such that $f\delta_0' = 0$.
9. Show that if $\Lambda \in \mathcal{D}'(\mathbb{R})$ is compactly supported, then Λ' is also compactly supported.
10. Verify that
$$\langle \Lambda, \varphi \rangle = \sum_{n=1}^{\infty} \varphi^{(n)}(n)$$
defines a distribution on $\mathbb{R}$. Is Λ compactly supported?
11. Let $H = \chi_{(-\infty,0)}$ and let h_n be a sequence of differentiable functions such that $h_n \to H$ in $\mathcal{D}'(\mathbb{R})$. Show that $h_n' \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$. Does the conclusion remain valid if $H = \chi_{(-\infty,0]}$?
12. Let $\Lambda_n \in \mathcal{D}'(\mathbb{R})$ be defined by
$$\langle \Lambda_n, \varphi \rangle = n\Big(\varphi\big(\tfrac{1}{n}\big) - \varphi\big(-\tfrac{1}{n}\big)\Big).$$
Determine $\lim \Lambda_n$.
13. For $a > 0$, define
$$\langle \Lambda_a, \varphi \rangle = \left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right) \frac{\varphi(x)}{|x|}\, dx + \int_{-a}^{a} \frac{\varphi(x) - \varphi(0)}{|x|}\, dx.$$

Show that Λ_a defines a distribution on $\mathcal{D}(\mathbb{R})$. Find $\lim_{a\to 0}\Lambda_a$ in $\mathcal{D}'(\mathbb{R})$ and compute its distributional derivative.

14. For $\Lambda \in \mathcal{D}'(\mathbb{R})$, define
$$\langle G, \varphi\rangle = \int_{\mathbb{R}} \langle \Lambda, \varphi_y\rangle \, dy,$$
where for $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we set $\varphi_y(x) = \varphi(x, y)$. Show that $G \in \mathcal{D}'(\mathbb{R}^2)$.
15. Let $\Lambda_i \in \mathcal{D}'(\mathbb{R})$ for $i = 1, 2$ be such that
$$\langle \Lambda_1, \varphi\rangle = 0 \quad \Longleftrightarrow \quad \langle \Lambda_2, \varphi\rangle = 0.$$
Show that $\Lambda_1 = c\Lambda_2$ for some constant c.
16. If $\Lambda \in \mathcal{D}'(\mathbb{R})$ satisfies $\Lambda^k = 0$, prove that Λ is a polynomial of degree at most $k - 1$.
17. Let $\Omega = (0, \infty)$. Define
$$\langle \Lambda, \varphi\rangle = \sum_{n=1}^{\infty} \varphi^{(n)}\left(\tfrac{1}{n}\right), \quad \varphi \in \mathcal{D}(\Omega).$$
Show that Λ is a distribution of infinite order, and prove that Λ cannot be extended to a distribution on $\mathbb{R}$.
18. If $\Lambda \in \mathcal{D}'(\mathbb{R})$ has order N, show that $\Lambda = f^{(N+2)}$ in $\mathcal{D}'(\mathbb{R})$ for some continuous function f. If $\Lambda = \delta_0$, what are the possible choices for f?
19. For $k \in \mathbb{N}$, define $f_k = k\chi_{\left(\frac{1}{k},\frac{2}{k}\right)}$. Show that $f_k \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$. Furthermore, show that although $f_k^2(x) \to 0$ pointwise, the sequence f_k^2 does not converge in the sense of distributions.
20. Define
$$f(x) = \begin{cases} x^2, & x < 1, \\ x^2 + 2x, & 1 \le x \le 2, \\ 2x, & x \ge 2. \end{cases}$$
Find the distributional derivative of f.
21. Define
$$f(t) = \begin{cases} e^{-t}, & t > 0, \\ -e^{t}, & t < 0. \end{cases}$$
Show that $f'' = 2\delta_0' + f$. Deduce that the Fourier transform of f is
$$\hat{f}(x) = -\frac{2ix}{1 + x^2}.$$
22. If $H = \chi_{(-\infty,0)}$, show that
$$\begin{aligned} &(a) \quad H * \varphi(x) = \int_{-\infty}^{x} \varphi(t)\, dt, \\ &(b) \quad \delta_0' * H = \delta_0, \\ &(c) \quad 1 * \delta_0' = 0, \\ &(d) \quad 1 * (\delta_0' * H) = 1 * \delta_0 = 1, \\ &(e) \quad (1 * \delta_0') * H = 0. \end{aligned}$$
23. Let $\{x_k\}$ be a sequence of real numbers with $\lim |x_k| = \infty$. Show that $\delta_{(x - x_k)} \to 0$ in the sense of distributions.
24. Determine all $f, g \in C^\infty(\mathbb{R})$ such that $f\delta_0 + g\delta_0' = 0$.
25. Define
$$f(x) = \begin{cases} e^{-x}, & x \ge 0, \\ 1, & x < 0. \end{cases}$$
Show that the Fourier transform of f satisfies $(1 - ix)\hat{f} = \hat{H}$ in the sense of tempered distributions, where $H = \chi_{(-\infty,0)}$.
26. Find the distributional derivative of $f(x) = e^{x^2}\chi_{[0,1]}(x)$.

27. Suppose $f \in L^\infty(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} f(y)e^{-y^2}e^{2xy}\,dy = 0 \quad \forall x \in \mathbb{R}.$$

Prove that $f \equiv 0$.

28. Let Λ be a distribution on $\mathbb{R}$ such that $x^2\Lambda = 0$. Show that $\Lambda = c\delta_0 + d\delta_0'$ for some constants c, d.
29. For $n \in \mathbb{N}$, let $f_n = \chi_{[0,n)}$. Find $\lim_{n\to\infty} f_n'$ in the weak* topology of $\mathcal{D}(\mathbb{R})$.
30. Classify all continuous functions on $\mathbb{R}$ that define tempered distributions.