MA642: Real Analysis -1

(Assignment 3: Metric and Normed Linear Spaces)
January - April, 2025

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) It is possible that $\mathbb{R}^2$ can be written as countable union of connected paths.
   (b) The cardinality of set of all the polynomials on $\mathbb{R}$ such that complement of their zero set are connected is countable.
   (c) There exists a non-empty open and connected set $A \subset \mathbb{R}^n$ such that every real valued function on A is continuous.
   (d) If a metric space X is path connected, then there exists a continuous function $f : [0, 1] \to X$ which is onto.
   (e) Let $f : (X, d) \to \mathbb{R}$ be such that $G_f = \{(x, f(x)) : x \in X\}$ is connected. Does it imply X is connected?
   (f) There exists a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ such that the graph G_f is connected in $\mathbb{R}^2$ but $\text{int}(\overline{G}_f)$ is non-empty in $\mathbb{R}^2$.

2. Let A be a connected subset of a metric space X, and let B be an open and closed set in X such that $A \cap B \neq \emptyset$. Show that $A \subset B$.

3. If E is connected subset of metric space X and $E \subset A \cup B$, where A and B are disjoint open subsets of X. Show that either $E \subset A$ or $E \subset B$.

4. Prove that $E \subset X$ is disconnected if and only if there exists non-empty open sets A and B such that $E = A \cup B$, where $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$.

5. If every pair points in X is contained in some connected set, show that X itself is connected.

6. If E and F are connected subsets of X and $E \cap F \neq \emptyset$, show that $E \cup F$ is connected.

7. If E and F are non-empty subsets of X such that $E \cup F$ is connected, show that $\bar{E} \cap \bar{F} \neq \emptyset$.

8. Show that the complement of any countable set E in $\mathbb{R}^2$ is path connected.

9. Prove that X is disconnected if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(\{0\}) = \emptyset$, while $f^{-1}((-\infty, 0)) \neq \emptyset$ and $f^{-1}((0, \infty)) \neq \emptyset$.

10. If X is connected and has at leat two point, show that X is uncountable.

11. Let $f : [a, b] \to \mathbb{R}$ be satisfying intermediate value property, and $f^{-1}(\{y\})$ is closed for every $y \in \mathbb{R}$. Show that f is continuous.

12. If $f : [a, b] \to [a, b]$ is continuous, show that f has a fixed point.

13. Let $f : [0, 2] \to \mathbb{R}$ be continuous and $f(0) = f(2)$. Show that there exists $x \in [0, 1]$ such that $f(x) = f(x + 1)$.

14. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and open, show that f is strictly monotone.

15. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and one to one, show that f is strictly monotone.

16. Prove that there does not exists continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q}$ and $f(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{Q}$.

17. If A and B are closed subsets of X such that $A \cup B$ and $A \cap B$ are connected, show that A and B both are connected.

18. Let $I = (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$ and $Q = \mathbb{Q} \cap [0, 1]$. Show that there exists a continuous map from I onto Q, but thee does not exist a continuous map from $[0, 1]$ to Q.

19. Suppose $f : \mathbb{R} \to \mathbb{R}$ is satisfying IVP. If G_f is closed, show that f is continuous.

20. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, show that f' satisfies has intermediate value property (IVP).

21. Justify that $\{(x, y) \in \mathbb{R}^2 : x^2 + y^3 \in \mathbb{R} \setminus \mathbb{Q}\}$ is a disconnected subset of $\mathbb{R}^2$ (with the usual topology).

22. Show that $GL_n(\mathbb{C})$ is path connected by using the fact that every polynomial on $\mathbb{C}$ has finitely many zeros. Does the set $GL_n(\mathbb{C})$ is open in $M_n(\mathbb{C})$?

23. Let $A \in GL_n(\mathbb{C})$. Show that the set $E = \left\{B \in L_n(\mathbb{C}) : \|B - A\| < \frac{1}{2\|A^{-1}\|}\right\}$ is open in $GL_n(\mathbb{C})$. And hence reduce that E is path connected in $L_n(\mathbb{C})$.