Assignment 3

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous m-a.e. on $\mathbb{R}$, then there must exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ m-a.e. on $\mathbb{R}$.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and if $g : \mathbb{R} \to \mathbb{R}$ is such that $f = g$ m-a.e. on $\mathbb{R}$, then g must be continuous m-a.e. on $\mathbb{R}$.
   (c) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous such that $f = g$ m-a.e. on $\mathbb{R}$, then it is necessary that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

2. If $(X, \mathcal{A})$ is a measurable space and $A \subset X$, then show that $\chi_A : X \to \mathbb{R}$ is $\mathcal{A}$-measurable iff A is $\mathcal{A}$-measurable.

3. If $(X, \mathcal{A})$ is a measurable space, then show that $f : X \to [-\infty, +\infty]$ is $\mathcal{A}$-measurable iff $\{x \in X : f(x) > r\} \in \mathcal{A}$ for each $r \in \mathbb{Q}$.

4. Let $f_n,\ f$ be real valued measurable functions on $\mathbb{R}$. Let $E = \{x \in \mathbb{R} :\ \lim f_n(x) = f(x)\}$. Show that E is Lebesgue measurable.

5. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} f(x) & \text{if } |f(x)| \le 5, \\ 0 & \text{if } |f(x)| > 5. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

6. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} 0 & \text{if } f(x) \in \mathbb{Q}, \\ 1 & \text{if } f(x) \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

7. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} -2 & \text{if } f(x) < -2, \\ f(x) & \text{if } -2 \le f(x) \le 3, \\ 3 & \text{if } f(x) > 3. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

8. Let $f : [0,\ 1] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0. \end{cases}$$
   Find the measure of the set $\{x \in \mathbb{R} :\ f(x) \ge 0\}$.
9. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. If $g : \mathbb{R} \to \mathbb{R}$ is continuous, then show that $g \circ f$ is $\mathcal{A}$-measurable.

10. Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f = \chi_{[0,1]}$ m-a.e. on $\mathbb{R}$? Justify.

11. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. If G is an open subset of $\mathbb{R}^2$, then show that $\{x \in X : (f(x), g(x)) \in G\}$ is $\mathcal{A}$-measurable.

12. If $f : \mathbb{R} \to \mathbb{R}$ is continuous m-a.e. on $\mathbb{R}$, then show that f is Lebesgue measurable.

13. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, then show that $f' : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x,\ .)$ and $f(.\ , y)$ are continuous then f is Lebesgue measurable.

15. Let E be a compact subset of $\mathbb{R}$. If $O_n = \left\{x \in \mathbb{R} : d(x, E) < \frac{1}{n}\right\}$, then $m(E) = \lim m(O_n)$. Does this conclusion true for E a bounded measurable set or unbounded closed set?

16. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$ and let $f : X \to \mathbb{R}$ be measurable. Let $A_n = \{x \in X : |f(x)| > n\}$. Show that A_n is $\mathcal{A}$-measurable and $\lim \mu(A_n) = 0$.

17. Let $f : [a, b] \to \mathbb{R}$ be Lebesgue measurable. Let $N = \{x \in [a, b] : f(x) = 0\}$. Show that $g = \chi_N + \frac{1}{f}\chi_{N^c}$ is Lebesgue measurable.

18. Let $f : \mathbb{R} \to \mathbb{R}$. Suppose for each $\epsilon > 0$ there exists an open set O such that $m(O) < \epsilon$ and f is constant on $\mathbb{R} \setminus O$. Show that f is Lebesgue measurable.

19. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous one-one and onto map. Then f sends Borels onto Borel sets.

20. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then a Lebesgue measurable $E \subset [a, b]$ satisfies $m(E) = 0$ implies $m(f(E)) = 0$ if and only if for every Lebesgue measurable subset $A \subset [a, b]$ the set $f(A)$ is Lebesgue measurable.