Assignment 3: Measure and Integration.

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) There exists an unbounded subset A of $\mathbb{R}$ such that $m^*(A) = 5$.
   (b) There exists an open subset A of $\mathbb{R}$ such that $[\frac{1}{2}, \frac{3}{4}] \subset A$ and $m(A) = \frac{1}{4}$.
   (c) There exists an open subset A of $\mathbb{R}$ such that $m(A) < \frac{1}{5}$ but $A \cap (a, b) \neq \emptyset$ for all $a, b \in \mathbb{R}$ with $a < b$.
   (d) If A and B are open subsets of $\mathbb{R}$ such that $A \subsetneq B$, then it is necessary that $m(A) < m(B)$.
   (e) A subset E of $\mathbb{R}$ is Lebesgue measurable iff $m^*(A \cup B) = m^*(A) + m^*(B)$ for each $A \subset E$ and for each $B \subset \mathbb{R} \setminus E$.
   (f) If $f : \mathbb{R} \to \mathbb{R}$ is continuous a.e. on $\mathbb{R}$, then there must exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ a.e. on $\mathbb{R}$.
   (g) If $g : \mathbb{R} \to \mathbb{R}$ is continuous and if $f : \mathbb{R} \to \mathbb{R}$ is such that $f = g$ a.e. on $\mathbb{R}$, then f must be continuous a.e. on $\mathbb{R}$.
   (h) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous such that $f = g$ a.e. on $\mathbb{R}$, then it is necessary that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

2. Let $f : [0, 2) \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1, \\ 3 - x & \text{if } 1 < x < 2. \end{cases}$
   Find $m^*(A)$, where $A = f^{-1}((\frac{9}{16}, \frac{5}{4})) = \{x \in [0, 2) : f(x) \in (\frac{9}{16}, \frac{5}{4})\}$.

3. Let $B \subset A \subset \mathbb{R}$ such that $m^*(B) = 0$. Show that $m^*(A \setminus B) = m^*(A)$.

4. Let $A \subset \mathbb{R}$ such that $m^*(A) > 0$. Show that there exists $B \subset A$ such that B is bounded and $m^*(B) > 0$.

5. If $A \subset \mathbb{R}$, then show that $m^*(A) = \inf\{m(G) : A \subset G,\ G \text{ is an open set in } \mathbb{R}\}$.

6. Let $E = \{x \in [0, 1] :$ The decimal representation of x does not contain the digit $5\}$. Show that $m(E) = 0$.

7. Let $A_n \subset \mathbb{R}$ for $n = 1, 2, ...$ such that $\sum\limits_{n=1}^{\infty} m^*(A_n) < \infty$.
   If $E = \{x \in \mathbb{R} : x \in A_n \text{ for infinitely many } n\}$, then show that $m(E) = 0$.

8. If G is a nonempty open subset of $\mathbb{R}$, then show that $m(G) > 0$.

9. Show that a subset E of $\mathbb{R}$ is Lebesgue measurable iff $m^*(I) = m^*(I \cap E) + m^*(I \setminus E)$ for every bounded open interval I of $\mathbb{R}$.

10. Let $A \subset E \subset B \subset \mathbb{R}$ such that A, B are Lebesgue measurable and $m(A) = m(B) < \infty$. Show that E is Lebesgue measurable.
    More generally, let $A \subset B \subset \mathbb{R}$ such that A is Lebesgue measurable and $m^*(B) = m(A) < \infty$. Show that B is Lebesgue measurable.

11. Let $A, B \subset \mathbb{R}$ such that $m^*(A) = 0$ and $A \cup B$ is Lebesgue measurable. Show that B is Lebesgue measurable.

12. Let $A, B \subset \mathbb{R}$ such that A is Lebesgue measurable and $m^*(A \triangle B) = 0$. Show that B is Lebesgue measurable.

13. Let $A \subset \mathbb{R}$ such that $A \cap B$ is Lebesgue measurable for every bounded subset B of $\mathbb{R}$. Show that A is Lebesgue measurable.

14. If E is a Lebesgue measurable subset of $\mathbb{R}$ and if $x \in \mathbb{R}$, then show that $E + x$ is Lebesgue measurable.

15. Let A be a countable subset of $\mathbb{R}$ and let $B \subset \mathbb{R}$ such that $m^*(B) = 0$. Show that $m^*(A+B) = 0$.

16. If E and F are Lebesgue measurable subsets of $\mathbb{R}$, then show that $m(E \cup F) + m(E \cap F) = m(E) + m(F)$.
More generally, let E be a Lebesgue measurable subset of $\mathbb{R}$ and let $A \subset \mathbb{R}$. Show that $m^*(E \cap A) + m^*(E \cup A) = m^*(E) + m^*(A)$.

17. Let I and J be disjoint open intervals in $\mathbb{R}$ and let $A \subset I$, $B \subset J$. Show that $m^*(A \cup B) = m^*(A) + m^*(B)$.

18. Let $A \subset [0, 1]$ be Lebesgue measurable with $m(A) = 1$. If $B \subset [0, 1]$, then show that $m^*(A \cap B) = m^*(B)$.

19. Let $E_i \subset (0, 1)$ $(i = 1, ..., n)$ be Lebesgue measurable sets such that $\sum_{i=1}^{n} m(E_i) > n - 1$. Show that $m(\cap_{i=1}^{n} E_i) > 0$.

20. If $A \subset \mathbb{R}$, then show that there exists a Lebesgue measurable subset E of $\mathbb{R}$ such that $m^*(A) = m(E)$.

21. Let $A \subset \mathbb{R}$ such that $m^*(A) > 0$. Show that there exist $x, y \in A$ such that $x - y \in \mathbb{R} \setminus \mathbb{Q}$.

22. Let A and B be Lebesgue measurable subsets of $(0, 1)$ such that $m(A) > \frac{1}{2}$ and $m(B) > \frac{1}{2}$. Prove that there exist $a \in A$ and $b \in B$ such that $a + b = 1$.

23. Let A be an unbounded Lebesgue measurable subset of $\mathbb{R}$ such that $m(A) < \infty$. Show that for each $\varepsilon > 0$, there exists a bounded Lebesgue measurable set B in $\mathbb{R}$ such that $B \subset A$ and $m(A \setminus B) < \varepsilon$.

24. Show that the Borel σ-algebra on $\mathbb{R}$ is generated by the class $\{(-\infty, x] : x \in \mathbb{Q}\}$.

25. Let $A \subset \mathbb{R}$ such that $m^*(A) = 0$. Show that $m^*(\{x^2 : x \in A\}) = 0$.

26. Let $A, B \subset \mathbb{R}$ such that $A \cup B$ is Lebesgue measurable and $m(A \cup B) = m^*(A) + m^*(B) < \infty$. Show that both A and B are Lebesgue measurable.

27. Examine whether $\mathcal{F}$ is a σ-algebra of subsets of $\mathbb{R}$, where
(a) $\mathcal{F} = \{A \subset \mathbb{R} : m^*(A) = 0 \text{ or } m^*(\mathbb{R} \setminus A) = 0\}$.
(b) $\mathcal{F} = \{A \subset \mathbb{R} : m^*(A) < \infty \text{ or } m^*(\mathbb{R} \setminus A) < \infty\}$.
(c) $\mathcal{F} = \{A \subset \mathbb{R} : A \text{ or } \mathbb{R} \setminus A \text{ is an open subset of } \mathbb{R}\}$.

28. Let X be an uncountable set. Show that $\{E \subset X : E \text{ is countable or } X \setminus E \text{ is countable}\}$ is a σ-algebra of subsets of X and that it is generated by the class $\{\{x\} : x \in X\}$.

29. Examine whether μ is an/a outer measure/measure on $\mathbb{R}$, where for each $A \subset \mathbb{R}$,

(a) $\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \neq \emptyset. \end{cases}$

(b) $\mu(A) = \begin{cases} 0 & \text{if } A \text{ is bounded}, \\ 1 & \text{if } A \text{ is unbounded}. \end{cases}$

30. If $A \subset \mathbb{R}$, then show that χ_A is a Lebesgue measurable function iff A is a Lebesgue measurable set.

31. Let E be a Lebesgue measurable subset of $\mathbb{R}$. Show that $f : E \to \mathbb{R}$ is Lebesgue measurable iff $\{x \in E : f(x) > r\}$ is Lebesgue measurable for each $r \in \mathbb{Q}$.

32. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$ be a Lebesgue measurable function. For each $x \in E$, let $g(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq 5, \\ 0 & \text{if } |f(x)| > 5. \end{cases}$
Show that $g : E \to \mathbb{R}$ is Lebesgue measurable.

33. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$ be a Lebesgue measurable function. For each $x \in E$, let $g(x) = \begin{cases} 0 & \text{if } f(x) \in \mathbb{Q}, \\ 1 & \text{if } f(x) \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Show that $g : E \to \mathbb{R}$ is Lebesgue measurable.

34. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$ be a Lebesgue measurable function. For each $x \in E$, let $g(x) = \begin{cases} -2 & \text{if } f(x) < -2, \\ f(x) & \text{if } -2 \leq f(x) \leq 3, \\ 3 & \text{if } f(x) > 3. \end{cases}$
Show that $g : E \to \mathbb{R}$ is Lebesgue measurable.

35. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$ be a Lebesgue measurable function. If $g : \mathbb{R} \to \mathbb{R}$ is continuous, then show that $g \circ f$ is Lebesgue measurable.

36. Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f = \chi_{[0,1]}$ a.e. on $\mathbb{R}$? Justify.

37. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$, $g : E \to \mathbb{R}$ be Lebesgue measurable functions. If G is an open subset of $\mathbb{R}^2$, then show that $\{x \in E : (f(x), g(x)) \in G\}$ is a Lebesgue measurable subset of $\mathbb{R}$.

38. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Show that $f' : [a, b] \to \mathbb{R}$ is Lebesgue measurable.

39. For each $x \in [0, 1]$, let $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ for some } m, n \in \mathbb{N} \text{ with g.c.d.}(m, n) = 1, \\ 0 & \text{otherwise}. \end{cases}$
Evaluate the Lebesgue integral $\int\limits_{[0,1]} f$.

40. For each $x \in [0,1]$, let $f(x) = \begin{cases} x^2 & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ x^3 & \text{if } x = \frac{1}{3^n} \text{ for some } n \in \mathbb{N}, \\ x^4 & \text{otherwise.} \end{cases}$

Evaluate the Lebesgue integral $\int\limits_{[0,1]} f$.

41. Let $f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in [0, \frac{1}{2}] \setminus C, \\ \cos(\pi x) & \text{if } x \in (\frac{1}{2}, 1] \setminus C, \\ x^2 & \text{if } x \in C. \end{cases}$

(C denotes the Cantor set.) Evaluate the Lebesgue integral $\int\limits_{[0,1]} f$.

42. Evaluate the Lebesgue integral $\int\limits_{[0,\infty)} e^{-[x]}\, dx$.

43. Let $f(x) = \begin{cases} e^{[x]} & \text{if } x \in \mathbb{Q}, \\ e^{-[x]} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Evaluate the Lebesgue integral $\int\limits_{(0,\infty)} f$.

44. Let $f(x) = \begin{cases} e^{|x|} & \text{if } x \in \mathbb{Q}, \\ e^{-|x|} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Evaluate the Lebesgue integral $\int\limits_{\mathbb{R}} f$.

45. Evaluate the Lebesgue integral $\int\limits_{(0,1]} \frac{1}{\sqrt[3]{x}}\, dx$.

46. Let $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1, \\ \frac{1}{x} & \text{if } x > 1. \end{cases}$

Evaluate the Lebesgue integral $\int\limits_{(0,\infty)} f$.

47. Evaluate the following:

(a) $\lim\limits_{n\to\infty} \int\limits_{-2}^{2} \frac{x^{2n}}{1+x^{2n}}\, dx$

(b) $\lim\limits_{n\to\infty} \int\limits_{[0,1]} \frac{1+nx}{(1+x)^n}\, dx$

(c) $\int\limits_{0}^{1} (\sum\limits_{n=1}^{\infty} \frac{x^n}{n})\, dx$

(d) $\lim\limits_{n\to\infty} \int\limits_{1}^{\infty} \frac{1}{1+x^{2n}}\, dx$

(e) $\sum\limits_{n=0}^{\infty} \int\limits_{0}^{1} \frac{x^2}{(1+x^2)^n}\, dx$

48. For any measure space, if $f \in L^1 \cap L^\infty$, then show that $f \in L^p$ for each $p \in (1, \infty)$.

49. For any measure space, show that $L^q \subset L^p + L^r$ if $0 < p < q < r \leq \infty$.