MA15010H: Multi-variable Calculus

(Assignment 3 Hint/model solutions: Directional derivatives and differentiability)
September - November, 2025

Problem 0.1. Let S be a nonempty open subset of $\mathbb{R}^2$ and let $f : S \to \mathbb{R}$ be such that the partial derivatives f_x and f_y exist at each point of S. If $f_x : S \to \mathbb{R}$ and $f_y : S \to \mathbb{R}$ are bounded, then show that f is continuous.

solution 0.2. *Since f_x and f_y are bounded, there exist $M_1, M_2 > 0$ such that $|f_x(x, y)| \leq M_1$ and $|f_y(x, y)| \leq M_2$ for all $(x, y) \in S$. Let $(x_0, y_0) \in S$. Since S is open in $\mathbb{R}^2$, there exists $r > 0$ such that $B_r((x_0, y_0)) \subseteq S$. For all $h, k \in \mathbb{R}$ with $|h| < \frac{r}{2},\ |k| < \frac{r}{2}$, we have*

$$\begin{aligned}
|f(x_0 + h, y_0 + k) - f(x_0, y_0)| &= |f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) \\
&\quad + f(x_0, y_0 + k) - f(x_0, y_0)| \\
&\leq |f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)| \\
&\quad + |f(x_0, y_0 + k) - f(x_0, y_0)| \\
&\leq |h||f_x(x_0 + \theta_1 h, y_0 + k)| + |k||f_y(x_0, y_0 + \theta_2 k)|
\end{aligned}$$

for some $\theta_1, \theta_2 \in (0, 1)$ (using Lagrange's mean value theorem of single real variable). Hence if $\epsilon > 0$, then choosing $\delta = \min\left\{\frac{r}{2}, \frac{\epsilon}{M_1+M_2}\right\} > 0$, we find that $|f(x_0 + h, y_0 + k) - f(x_0, y_0)| \leq M_1|h| + M_2|k| < \epsilon$ for all $(h, k) \in \mathbb{R}^2$ with $\|(h, k)\| = \sqrt{h^2 + k^2} < \delta$. Therefore f is continuous at (x_0, y_0). Since $(x_0, y_0) \in S$ is arbitrary, f is continuous.

Problem 0.3. Find all $u \in \mathbb{R}^2$ with $\|u\| = 1$ for which the directional derivative $D_u f(0, 0)$ exists (in $\mathbb{R}$), if for all $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = \begin{cases} 1, & \text{if } y < x^2 < 2y, \\ 0, & \text{otherwise.} \end{cases}$$

solution 0.4. *Let $u = (u_1, u_2) \in \mathbb{R}^2$ with $\|u\| = 1$. We have*

$$\lim_{t\to 0} \frac{f((0, 0) + tu) - f(0, 0)}{t} = \lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} = \lim_{t\to 0} \frac{0}{t} = 0.$$

(The inequalities $tu_2 < t^2u_1^2 < 2tu_2$ are equivalent to the inequalities:

$$(i) u_2 < tu_1^2 < 2u_2 \text{ if } t > 0$$
$$(ii) u_2 > tu_1^2 > 2u_2 \text{ if } t < 0.$$

We can make $|tu_1^2|$ arbitrarily small for sufficiently small $|t| > 0$ and hence for such t, at least one inequality in each of (i) and (ii) cannot be satisfied. Thus we get $f(tu_1, tu_2) = 0$ for sufficiently small $|t| > 0$.) Therefore $D_u f(0, 0)$ exists (and equals 0) for each $u \in \mathbb{R}^2$ with $\|u\| = 1$.

Problem 0.5. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that all the directional derivatives of f at $(0,0)$ exist (in $\mathbb{R}$), then f must be differentiable at $(0,0)$.

solution 0.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$f(x,y) = \begin{cases} \frac{x^2y\sqrt{x^2+y^2}}{x^4+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

We know that f is continuous at each point of $\mathbb{R}^2 \setminus \{(0,0)\}$. Let $\epsilon > 0$. We have

$$|f(x,y) - f(0,0)| = \left| \frac{x^2y}{x^4+y^2} \right| \sqrt{x^2+y^2} \leq \frac{1}{2}\sqrt{x^2+y^2}$$

for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $|f(x,y) - f(0,0)| = 0$ if $(x,y) = (0,0)$. Hence choosing $\delta = 2\epsilon > 0$, we find that $|f(x,y) - f(0,0)| < \epsilon$ for all $(x,y) \in \mathbb{R}^2$ satisfying $\|(x,y) - (0,0)\| = \sqrt{x^2+y^2} < \delta$. This shows that f is continuous at $(0,0)$ and therefore f is continuous.

If $u = (u_1, u_2) \in \mathbb{R}^2$ with $\|u\| = 1$, then

$$\lim_{t \to 0} \frac{f((0,0)+tu) - f(0,0)}{t} = \lim_{t \to 0} \frac{u_1^2 u_2 t|t|\sqrt{u_1^2+u_2^2}}{t^2u_1^4 + u_2^2} = 0$$

i.e., $D_u f(0,0)$ exists. Hence all the directional derivatives of f at $(0,0)$ exist.

Again,

$$\lim_{(h,k) \to (0,0)} \frac{|f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim_{(h,k) \to (0,0)} \frac{h^2k}{h^4+k^2} \neq 0.$$

since $\left(\frac{1}{n}, \frac{1}{n^2}\right) \to (0,0)$, but

$$\frac{\frac{1}{n^2}\frac{1}{n^2}}{\frac{1}{n^4} + \frac{1}{n^4}} = \frac{1}{2} \neq 0.$$

Hence f is not differentiable at $(0,0)$. Therefore the given statement is ***FALSE****.*

Problem 0.7. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, if for all $(x,y) \in \mathbb{R}^2$,

$$f(x,y) = \begin{cases} x^{4/3}\sin\left(\frac{y}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

solution 0.8. *Let $E = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. Since*

$$f_x(x,y) = \frac{4}{3}x^{1/3}\sin\left(\frac{y}{x}\right) - \frac{y}{x^{2/3}}\cos\left(\frac{y}{x}\right) \quad \textit{and} \quad f_y(x,y) = x^{1/3}\cos\left(\frac{y}{x}\right)$$

for all $(x,y) \in E$. $f_x : E \to \mathbb{R}$ *and* $f_y : E \to \mathbb{R}$ *are continuous. Hence* f *is differentiable at all* $(x,y) \in E$. *Let* $y_0 \in \mathbb{R}$ *and let* $\epsilon > 0$. *Then*

$$f_x(0,y_0) = \lim_{h\to 0} \frac{f(h,y_0) - f(0,y_0)}{h} = \lim_{h\to 0} h^{1/3}\sin(\frac{y_0}{h}) = 0$$

(since $|h^{1/3}\sin(\frac{y_0}{h})| \leq |h|^{1/3}$ *for all* $h \in \mathbb{R}\backslash\{0\}$*) and*

$$f_y(0,y_0) = \lim_{k\to 0} \frac{f(0,y_0+k) - f(0,y_0)}{k} = 0.$$

Also, for all $(x,y) \in E$, *we have* $f_y(x,y) = x^{1/3}\cos(y/x)$, *and so*

$$|f_y(x,y) - f_y(0,y_0)| \leq |x|^{1/3} < \epsilon \quad \textit{for all } (x,y) \in B_\delta((0,y_0)),$$

where $\delta = \epsilon^3 > 0$. *Thus* $f_x(0,y_0)$ *exists (in* $\mathbb{R}$*),* $f_y(x,y_)$ *exists (in* $\mathbb{R}$*) for all* $(x,y) \in \mathbb{R}^2$ *and* $f_y : \mathbb{R}^2 \to \mathbb{R}$ *is continuous at* $(0,y_0)$. *Hence by Ex.21 of Practice Problem Set - 3,* f *is differentiable at* $(0,y_0)$. *Therefore* f *is differentiable at all points of* $\mathbb{R}^2$.

Alternative solution: As shown above, f is differentiable at all $(x,y) \in \mathbb{R}^2$ for which $x \neq 0$. Let $y_0 \in \mathbb{R}$. Then as shown above, $f_x(0,y_0) = f_y(0,y_0) = 0$. For all $(h,k) \in \mathbb{R}^2$ with $h \neq 0$, we have

$$\begin{aligned}\epsilon(h,k) = \frac{|f(h,y_0+k) - f(0,y_0) - hf_x(0,y_0) - kf_y(0,y_0)|}{\sqrt{h^2+k^2}} &= \frac{h^{4/3}\sin|\left(\frac{y_0+k}{h}\right)|}{\sqrt{h^2+k^2}} \\ &= \frac{|h|^{1/3}|h||\sin(\frac{y_0+k}{h})|}{\sqrt{h^2+k^2}} \leq |h|^{1/3}.\end{aligned}$$

Also, $\epsilon(0,k) = 0$ for all $k \in \mathbb{R} \setminus \{0\}$. Hence it follows that

$$\lim_{(h,k)\to(0,0)} \epsilon(h,k) = 0.$$

Consequently, f is differentiable at $(0,y_0)$. Therefore f is differentiable at all points of $\mathbb{R}^2$.

Problem 0.9. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable at $x_0 \in S^0$ and let $f(x_0) = 0$. If $g : S \to \mathbb{R}$ is continuous at x_0, then show that $fg : S \to \mathbb{R}$, defined by $(fg)(x) = f(x)g(x)$ for all $x \in S$, is differentiable at x_0.

solution 0.10. *Since* f *is differentiable at* x_0, *there exists* $\alpha \in \mathbb{R}^m$ *such that*

$$\lim_{h\to 0} \frac{|f(x_0+h) - f(x_0) - \alpha \cdot h|}{\|h\|} = 0.$$

For all $h \in \mathbb{R}^m$ *for which* $x_0 + h \in S$, *we have*

$$(fg)(x_0+h) - (fg)(x_0) - g(x_0)\alpha\cdot h = (f(x_0+h) - f(x_0) - \alpha\cdot h)g(x_0+h) + (g(x_0+h) - g(x_0))\alpha\cdot h.$$

Hence for all $h \in \mathbb{R}^m \setminus \{0\}$ for which $x_0 + h \in S$, we have

$$\frac{|(fg)(x_0+h) - (fg)(x_0) - g(x_0)\alpha \cdot h|}{\|h\|} \leq \frac{|f(x_0+h) - f(x_0) - \alpha \cdot h|}{\|h\|}|g(x_0+h)| + |g(x_0+h) - g(x_0)|\frac{|\alpha \cdot h|}{\|h\|}.$$

Since g is continuous at x_0, $\lim_{h\to 0} g(x_0+h) = g(x_0)$ and since $|\alpha \cdot h| \leq \|\alpha\|\|h\|$, it follows that

$$\lim_{h\to 0} \frac{|(fg)(x_0+h) - (fg)(x_0) - g(x_0)\alpha \cdot h|}{\|h\|} = 0.$$

Since $g(x_0)\alpha \in \mathbb{R}^m$, we conclude that fg is differentiable at x_0.

Problem 0.11. Show that $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0, y_0) \in S^0$ if and only if there exist functions $\varphi, \psi : S \to \mathbb{R}$ such that φ, ψ are continuous at (x_0, y_0) and

$$f(x,y) - f(x_0,y_0) = (x - x_0)\varphi(x,y) + (y - y_0)\psi(x,y)$$

for all $(x, y) \in S$.

solution 0.12. *We first assume that f is differentiable at (x_0, y_0). Then $\alpha = f_x(x_0, y_0)$ and $\beta = f_y(x_0, y_0)$ exist (in $\mathbb{R}$). For each $(x, y) \in S$, let*

$$g(x,y) = f(x,y) - f(x_0,y_0) - \alpha(x - x_0) - \beta(y - y_0),$$

then define

$$\varphi(x,y) = \begin{cases} \alpha + \dfrac{(x-x_0)g(x,y)}{(x-x_0)^2 + (y-y_0)^2}, & (x,y) \neq (x_0,y_0), \\ \alpha, & (x,y) = (x_0,y_0), \end{cases}$$

and

$$\psi(x,y) = \begin{cases} \beta + \dfrac{(y-y_0)g(x,y)}{(x-x_0)^2 + (y-y_0)^2}, & (x,y) \neq (x_0,y_0), \\ \beta, & (x,y) = (x_0,y_0). \end{cases}$$

If $(x, y) \in S \setminus \{(x_0, y_0)\}$, then

$$(x - x_0)\varphi(x,y) + (y - y_0)\psi(x,y) = \alpha(x - x_0) + \beta(y - y_0) + g(x,y) = f(x,y) - f(x_0,y_0).$$

Also, if $(x, y) = (x_0, y_0)$, then

$$(x - x_0)\varphi(x,y) + (y - y_0)\psi(x,y) = 0 = f(x,y) - f(x_0,y_0).$$

Hence for all $(x, y) \in S$, $f(x, y) - f(x_0, y_0) = (x - x_0)\varphi(x, y) + (y - y_0)\psi(x, y)$.

Again, for all $(x, y) \in S \setminus \{(x_0, y_0)\}$, we have

$$|\varphi(x,y) - \varphi(x_0,y_0)| = \frac{|x-x_0||g(x,y)|}{(x-x_0)^2 + (y-y_0)^2} \leq \frac{|g(x,y)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}.$$

Since f is differentiable at (x_0, y_0), the limit

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|g(x,y)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0,$$

and hence it follows that

$$\lim_{(x,y)\to(x_0,y_0)} \varphi(x,y) = \varphi(x_0,y_0).$$

Therefore φ is continuous at (x_0,y_0). Similarly, we can show ψ is continuous at (x_0,y_0).

Conversely, let there exist functions $\varphi,\psi : S \to \mathbb{R}$ such that φ,ψ are continuous at (x_0,y_0) and

$$f(x,y) - f(x_0,y_0) = (x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y)$$

for all $(x,y) \in S$. Then for all $(x,y) \in S \setminus \{(x_0,y_0)\}$, we have

$$\begin{aligned}
&\frac{|f(x,y) - f(x_0,y_0) - (x-x_0)\varphi(x_0,y_0) - (y-y_0)\psi(x_0,y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \\
&\le \frac{(x-x_0)|\varphi(x,y) - \varphi(x_0,y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} + \frac{(y-y_0)|\psi(x,y) - \psi(x_0,y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \\
&\le |\varphi(x,y) - \varphi(x_0,y_0)| + |\psi(x,y) - \psi(x_0,y_0)|.
\end{aligned}$$

Since φ and ψ are continuous at (x_0,y_0),

$$\lim_{(x,y)\to(x_0,y_0)} |\varphi(x,y) - \varphi(x_0,y_0)| = 0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} |\psi(x,y) - \psi(x_0,y_0)| = 0.$$

Hence,

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y) - f(x_0,y_0) - (x-x_0)\varphi(x_0,y_0) - (y-y_0)\psi(x_0,y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0,$$

and therefore f is differentiable at (x_0,y_0).

Problem 0.13. Let the temperature $T(x,y)$ at any point $(x,y) \in \mathbb{R}^2$ be given by $T(x,y) = 2x^2 + xy + y^2$. An insect is at the point $(1,1)$.
(a) What is the best direction for the insect to move to feel cooler?
(b) In which direction should the insect move to feel no change in temperature?

solution 0.14. *Since $T_x(x,y) = 4x + y$ and $T_y(x,y) = x + 2y$ for all $(x,y) \in \mathbb{R}^2$, $T_x : \mathbb{R}^2 \to \mathbb{R}$ and $T_y : \mathbb{R}^2 \to \mathbb{R}$ are continuous and hence $T : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Since*

$$\nabla T(1,1) = (T_x(1,1), T_y(1,1)) = (5,3),$$

the temperature will decrease fastest in the direction

$$-\frac{1}{\|\nabla T(1,1)\|}\nabla T(1,1) = \left(-\frac{5}{\sqrt{34}}, -\frac{3}{\sqrt{34}}\right),$$

and so this is the best direction for the insect to start moving to feel cooler.

Again, if $u = (u_1,u_2) \in \mathbb{R}^2$ with $\|u\| = 1$ is the direction for the insect to feel no change in temperature, then we must have

$$D_u T(1,1) = \nabla T(1,1) \cdot u = 0.$$

This gives $5u_1 + 3u_2 = 0$. *Since we also have* $u_1^2 + u_2^2 = 1$, *we get*

$$u = \left(\frac{3}{\sqrt{34}}, -\frac{5}{\sqrt{34}}\right) \quad \textit{or} \quad \left(-\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right).$$