MA15010H: Multi-variable Calculus

(Assignment 3: Directional derivatives and differentiability)

September - November, 2025

1. Let S be a nonempty open subset of $\mathbb{R}^2$ and let $f : S \to \mathbb{R}$ be such that the partial derivatives f_x and f_y exist at each point of S. If $f_x : S \to \mathbb{R}$ and $f_y : S \to \mathbb{R}$ are bounded, then show that f is continuous.

2. Find all $u \in \mathbb{R}^2$ with $\|u\| = 1$ for which the directional derivative $D_u f(0,0)$ exists (in $\mathbb{R}$), if for all $(x, y) \in \mathbb{R}^2$,
$$f(x,y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$$

3. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that all the directional derivatives of f at $(0,0)$ exist (in $\mathbb{R}$), then f must be differentiable at $(0,0)$.

4. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, if for all $(x, y) \in \mathbb{R}^2$,
$$f(x,y) = \begin{cases} x^{4/3} \sin\left(\frac{y}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

5. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable at $x_0 \in S^0$ and let $f(x_0) = 0$. If $g : S \to \mathbb{R}$ is continuous at x_0, then show that $fg : S \to \mathbb{R}$, defined by $(fg)(x) = f(x)g(x)$ for all $x \in S$, is differentiable at x_0.

6. Show that $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0, y_0) \in S^0$ iff there exist functions $\varphi, \psi : S \to \mathbb{R}$ such that φ, ψ are continuous at (x_0, y_0) and $f(x,y) - f(x_0,y_0) = (x - x_0)\varphi(x,y) + (y - y_0)\psi(x,y)$ for all $(x, y) \in S$.

7. Let the temperature $T(x, y)$ at any point $(x, y) \in \mathbb{R}^2$ be given by $T(x,y) = 2x^2 + xy + y^2$. An insect is at the point $(1, 1)$.
   (a) What is the best direction for the insect to move to feel cooler?
   (b) In which direction should the insect move to feel no change in temperature?