Assignment 3

1. Let X and Y be two normed linear spaces and $T \in B(X, Y)$. Define the linear map $T^* : Y^* \to X^*$ by $T^*(f) = f \circ T$, for all $f \in Y^*$. Show that

   (a) $\ker T^* = (\mathrm{Im} T)^{\perp}$.

   (b) T is bijective implies T^* is bijective.

2. Let K be a compact set in $\mathbb{C}$. Then there exists an operator $T \in B\left(l^2(\mathbb{N})\right)$ such that $\sigma(T) = K$.

3. Let $T : L^2[0,1] \to L^2[0,1]$ be a linear map which is defined by

$$T(f)(x) = \int_0^x f(t)dt.$$

   Show that

   (a) T is bounded, compact and $\|T\| = \frac{2}{\pi}$. Does T invertible?

   (b) Define $\langle Tf, g\rangle = \langle f, T^*g\rangle$. Find the adjoint operator T^* of T.

   (b) Prove that spectral radius $r(T) = 0$ and $0 \in \sigma_c(T)$, continuous spectrum.

4. Let T be linear operator on a Hilbert space H such that $(Tx, y) = (x, Ty)$. Show that T is continuous.

5. Let $\{e_n : \ n \in \mathbb{N}\}$ be an orthonormal basis for a separable Hilbert space H. Define a linear map $T : H \to H$ by $T(e_n) = a_n e_n, \ n = 1, 2, \ldots$. Show that

   (a) T is bounded if and only if sequence $\{a_n\}$ is bounded.

   (b) T is compact if and only if $\lim\limits_{n\to\infty} a_n = 0$.

6. Let $g \in C[0,1]$ and $T : L^2[0,1] \to L^2[0,1]$ be a linear map which is defined by

$$T(f)(t) = g(t)f(t).$$

   Find the spectrum $\sigma(T)$ and deduce that T is not compact.

7. Let $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$. For $x = (x_k)_{-\infty}^{\infty} \in l^2(\mathbb{Z})$, define $T(x) = (x_{k-1})_{-\infty}^{\infty}$ (right shift operator). Show that

   (a) the point spectrum $\sigma_p(T) = \emptyset$,

   (b) $\mathrm{Im}(\lambda I - T) = l^2(\mathbb{Z})$ if $|\lambda| \neq 1$,

   (c) spectrum $\sigma(T) = \{\lambda : \ |\lambda| = 1\}$.

8. Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$. For $x \in l^2(\mathbb{N})$, define $T(x) = (x_2, x_3, \ldots)$. Prove that

   (a) $\rho(T) = \{\lambda \in \mathbb{C} : \ |\lambda| > 1\}$.

   (b) $\sigma_c((T) = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}$.

   (c) $\sigma_p((T) = \{\lambda \in \mathbb{C} : \ |\lambda| < 1\}$.

   (d) $\sigma_r(T) = \emptyset$.

9. Let g be a continuous and bounded function on $\mathbb{R}$. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be linear map defined by $T(f)(t) = g(t)f(t)$. Show that T is bounded and spectrum

$$\sigma(T) = \overline{\{g(x) : x \in \mathbb{R}\}}.$$

10. Let X be a Banach space and $T \in B(X)$. Prove that $\exp(T) = \sum_0^\infty \frac{T^n}{n!}$ is invertible and $\sigma(\exp T) = \exp(\sigma(T))$.

11. Let P be a bounded linear projection on a Hilbert space H. Show that $\sigma(P) = \sigma_p(P) = \{0, 1\}$ and for $\lambda \notin \{0, 1\}$, resolvent function for P is given by

$$R(\lambda) = (\lambda I - P)^{-1} = \frac{I}{\lambda} + \frac{1}{\lambda(1-\lambda)} P.$$

12. A bounded linear operator T on a separable Hilbert space H is called Hilbert-Schmidt operator if there exists an orthonormal basis $\{e_n : \ n \in \mathbb{N}\}$ such that $\sum \|Te_n\|^2 < \infty$. Write $\|T\|_{\text{H.S.}} = \left(\sum \|Te_n\|^2\right)^{1/2}$. Show that

    (a) T is a compact operator.

    (b) $\|T^*\|_{\text{H.S.}} = \|T\|_{\text{H.S.}}$.

    (c) Hilbert-Schmidt norm is independent of choice of orthonormal basis.

13. Let T be a normal operator on a Hilbert space H. Show that $\|T^2\| = \|T\|^2$.