Assignment 2

1. Show that every infinite-dimensional separable normed linear space contains a countable linearly independent dense subset.

2. Show that $\{(x_1, x_2, \ldots, x_n, \ldots) \in l_2(\mathbb{N}) : \ |x_n| \leq \frac{1}{n}\}$ is a compact convex subset of $l^2(\mathbb{N})$ with empty interior.

3. Let M be a closed proper subspace of a normed linear space X. Then M is nowhere dense in X.

4. Prove that an infinite dimensional Banach space X can not be expressed as the countable union of compact subsets of X.

5. Let X, Y be normed linear spaces and let $T : X \to Y$ be a linear map. If for every absolutely convergent series $\sum\limits_{n=1}^{\infty} x_n$ in X, $\sum\limits_{n=1}^{\infty} Tx_n$ is a convergent series in Y, then show that T is bounded.

6. For $1 \leq p \leq \infty$, define a linear operstor on $l^p(\mathbb{N})$ by $T(x) = \left(x_1, \dfrac{x_2}{2}, \dfrac{x_3}{3}, \ldots\right)$. Find $\|T\|$.

7. Let $X = (C[0,1], \|\cdot\|_\infty)$. For $f \in C[0,1]$ define $K(f)(t) = \int\limits_0^t f(s)ds$. Show that

   (a) K is one one but not onto.
   (b) For each $n \in \mathbb{N}$, the power of operator K satisfies $\|K^n\| = \frac{1}{n!}$.
   (c) Operator $T = I + K$ is invertible.

8. Let $X = (C[0,\pi], \|\cdot\|_\infty)$. For $f \in C[0,\pi]$ define $T(f)(x) = \int\limits_0^x \sin(x+y)f(y)dy$. Find $\|T\|$.

9. Let X and Y be two Banach spaces and $T : X \to Y$ is a continuous linear bijection. Then show that $\inf\{\|x\| : \|Tx\| = 1\} = \|T^{-1}\|$.

10. Let $\phi \in L^\infty(\mathbb{R})$. For $1 \leq p < \infty$, define an operator on $L^p(\mathbb{R}^n)$ by $M_\phi(f) = \phi f$. Show that $\|M_\phi\| = \|\phi\|_\infty$. Whether the conclusion is true for $p = \infty$?

11. Let M be a closed subspace of a normed linear space X. Show that the projection $\pi : \ X \to X/M$ defined by $\pi(x) = x + M$ is a continuous linear surjective open map with $\|\pi\| < 1$. If $M \subsetneq X$ then $\|\pi\| = 1$.

12. Let X be normed linear space. Then X is complete if and only if X/M and M both are complete.

13. Let $X = \{f \in C[0,1] : \ f(0) = 0\}$ and $M = \left\{ f \in X : \ \int\limits_0^1 f(t)dt = 0 \right\}$. Show that

(a) M is an infinite dimensional closed subspace of X.
(b) There does not exists any $f \in X$ with $\|f\|_\infty = 1$ such that $\text{dist}(f, M) \geq 1$.

14. Let $x = (x_1, x_2, \ldots) \in l^\infty(\mathbb{N})$. Then the norm of $\tilde{x} \in l^\infty(\mathbb{N})/c_o(\mathbb{N})$ can be expressed as $\|\tilde{x}\| = \lim\limits_{n\to\infty} \sup |x_n|$.

15. Let X and Y be two Banach spaces and $T \in B(X,Y)$. Show that followings are equivalent:
(a) T is injective and has closed range.
(b) There is $k \geq 0$ such that $\|x\| \leq k\|T(x)\|$ for all $x \in X$.

16. Let X and Y be two normed linear spaces and $T : X \to Y$ such that $\dim T(X) < \infty$ and $\ker T$ is closed. Then show that T is bounded.

17. Suppose X can be made Banach space with respect to norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If there exists $m > 0$ such that $\|x\|_1 \leq m\|x\|_2$ for all $x \in X$. Then both norms are equivalent.

18. Let X be a Banach space and Y be a normed linear space. Suppose $T_n \in B(X,Y)$ such that $\lim\limits_{n\to\infty} T_n(x)$ exists for each $x \in X$. Write $T(x) = \lim\limits_{n\to\infty} T_n(x)$. Show that T is bounded.

19. Let (a_n) be a sequence of real numbers such that for each $x \in l^2(\mathbb{N})$, the vector $(a_1x_1, a_2x_2, \ldots) \in l^2(\mathbb{N})$. Define an operator T on $l^2(\mathbb{N})$ by $T(x) = (a_1x_1, a_2x_2, \ldots)$. Show that T is bounded.

20. Let X and Y be two normed linear spaces and $T_n, T \in B(X,Y)$ such that $T_n \to T$. Suppose $x_n \to x$. Then show that $T_nx_n \to Tx$ in Y.

21. Let X be a Banach space and let $T \in B(X)$ with $\|T\| < 1$. Then show that $I - T$ is invertible and $(I-T)^{-1} = \sum\limits_{n=0}^{\infty} T^n \in B(X)$.

22. Let (T_n) be a sequence of bounded linear operator on a Banach space X such that $\|T_n - T\| \to 0$. If T_n^{-1} exists, $\forall\, n \in \mathbb{N}$ and $\|T_n^{-1}\| < 1$, then prove that $T^{-1} \in B(X)$.

23. Let X and Y be two normed linear spaces and $T \in B(X,Y)$ sends each open subset in X to an open subset in Y. Prove that T is onto.

24. Let $T : (C^1[0,1], \|\,.\,\|_\infty) \to (C[0,1], \|\,.\,\|_\infty)$ be a linear transformation defined by $Tf(s) = f'(s) + \int\limits_0^s f(t)dt$. Show that the graph of T is closed but T is not continuous.