MA746: Fourier Analysis

(Assignment 2: Fourier Transform)
July – November, 2025

1. (a) Let $f \in C_c^\infty(\mathbb{R})$ be nonzero and let P be a polynomial of degree $n \geq 1$. Determine whether the function $P\hat{f}$ is bounded on $\mathbb{R}$.
   (b) Is the subspace
$$\{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \text{ is compact}\}$$
   dense in $L^2(\mathbb{R})$?
2. Suppose f is continuously differentiable on $[-R, R]$. Prove that there exists a constant $C > 0$ such that
$$|\hat{f}(\xi)| \leq \frac{C}{|\xi|}, \qquad \xi \neq 0.$$
3. Let $f, g \in L^2(\mathbb{R})$. Show that the convolution $f * g$ is a bounded continuous function on $\mathbb{R}$, and that
$$\lim_{|x|\to\infty} (f * g)(x) = 0.$$
4. Let $f \in L^1(\mathbb{R})$ satisfy $f(x) > 0$ for all $x \in \mathbb{R}$. Prove that there exists $\delta > 0$ such that
$$|\hat{f}(\xi)| < \hat{f}(0), \qquad |\xi| > \delta.$$
5. For $n \in \mathbb{N}$, define
$$F_n(x) = \chi_{[-1,1]} * \chi_{[-n,n]}(x).$$
   Verify that $F_n \in C_c(\mathbb{R})$ with $\|F_n\|_\infty = 2$. Does the sequence $\{F_n(x)\}$ converge uniformly to 2 on $\mathbb{R}$?
6. For $1 \leq p < \infty$, let $f \in L^p(\mathbb{R})$ and set
$$F(x) = \int_x^{x+1} f(t)\, dt.$$
   Show that $F \in C_0(\mathbb{R})$. Does this conclusion remain valid for $f \in L^\infty(\mathbb{R})$?
7. For $f \in L^1(\mathbb{R})$, prove the identity
$$2\hat{f}(\xi) = \int_{\mathbb{R}} \left[f(x) - f\left(x - \tfrac{\pi}{\xi}\right) \right] e^{-i\xi x}\, dx,$$
   and deduce the Riemann–Lebesgue lemma.
8. Let $f, g \in L^1(\mathbb{R})$. Prove that
$$\int_{\mathbb{R}} f(y)\hat{g}(y)\, dy = \int_{\mathbb{R}} \hat{f}(\xi) g(\xi)\, d\xi.$$
   If $\hat{f} \in L^1(\mathbb{R})$, deduce the Fourier inversion formula for f.
9. For $n \in \mathbb{N}$, define
$$f(x) = \frac{x^n}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$
   Show that
$$\hat{f}(\xi) = P_n(\xi) e^{-\frac{\xi^2}{2}},$$
   where P_n is a polynomial of degree n.
10. A continuous function $f : \mathbb{R} \to \mathbb{C}$ is of **moderate decrease** if there exists $A > 0$ such that
$$|f(x)| \leq \frac{A}{1 + x^2}, \qquad x \in \mathbb{R}.$$
   Suppose f is of moderate decrease and satisfies
$$\int_{\mathbb{R}} f(y) e^{-y^2} e^{2xy}\, dy = 0 \quad \forall x \in \mathbb{R}.$$

Prove that $f \equiv 0$.

11. Let f be of moderate decrease and define
$$f * K_\lambda(x) = \tfrac{1}{2\pi} \int_{-\lambda}^{\lambda} \left(1 - \tfrac{|\xi|}{\lambda}\right) \hat{f}(\xi) e^{i\xi x} \, d\xi.$$
Show that $f * K_\lambda \to f$ uniformly as $\lambda \to \infty$.
12. Let $\{k_\lambda\} \subset L^1(\mathbb{R})$ be a family of good kernels. If $f \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, prove that $f * k_\lambda \to f$ uniformly on every compact subset of $\mathbb{R}$.
13. For $1 \leq p \leq 2$, prove that
$$\{f \in L^p(\mathbb{R}) : \operatorname{supp} \hat{f} \text{ compact}\}$$
is dense in $L^p(\mathbb{R})$.
14. Show that
$$X = \{\hat{f} : f \in L^1(\mathbb{R})\}$$
is dense in $C_0(\mathbb{R})$.
15. Let $f \in C_c^2(\mathbb{R})$. Prove that there exists $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\hat{g} = f$.
16. For $f \in L^2(\mathbb{R})$, define the translation operator $\tau_x f(y) = f(y - x)$. Show that
$$X = \{\tau_x f : x \in \mathbb{R}\}$$
is dense in $L^2(\mathbb{R})$ if and only if $\hat{f}(\xi) \neq 0$ almost everywhere.
17. Let $f \in L^1(\mathbb{R})$ with compact support. Prove that $\hat{f}$ is real-analytic on $\mathbb{R}$. Does $\hat{f} \in L^1(\mathbb{R})$? What additional conclusion holds if $f \in C_c^2(\mathbb{R})$?
18. Let $f \in L^1(\mathbb{R})$ with $f \geq 0$. Show that
$$\|\hat{f}\|_\infty = \hat{f}(0) = \|f\|_1.$$
19. Suppose $f \in L^1(\mathbb{R})$ is continuous at 0 and $\hat{f}(\xi) \geq 0$ for all ξ. Prove that $\hat{f} \in L^1(\mathbb{R})$ and
$$f(0) = \int_{\mathbb{R}} \hat{f}(\xi) \, d\xi.$$
20. For $n \in \mathbb{N}$, let $g_n = \chi_{[-1,1]} * \chi_{[-n,n]}$. Show that g_n is the Fourier transform of
$$f_n(x) = \frac{\sin x \sin nx}{\pi^2 x^2} \in L^1(\mathbb{R}),$$
and that $\|f_n\|_1 \to \infty$. Conclude that the Fourier transform maps $L^1(\mathbb{R})$ into a proper subspace of $C_0(\mathbb{R})$.
21. For $f \in L^1(\mathbb{R})$, define $f_\lambda(x) = \lambda f(\lambda x)$ and
$$\varphi_\lambda(t) = 2\pi \sum_{j=-\infty}^{\infty} f_\lambda(t + 2\pi j).$$
Show that
$$\lim_{\lambda \to \infty} \|\varphi_\lambda\|_{L^1(S^1)} = \|f\|_{L^1(\mathbb{R})}.$$
22. For $f \in L^1(\mathbb{R})$, define
$$g(t) = 2\pi \sum_{n=-\infty}^{\infty} f(t + 2\pi n).$$
Show that g is periodic and
$$\|g\|_{L^1(S^1)} \leq \|f\|_{L^1(\mathbb{R})}.$$
23. For $1 \leq p < \infty$, suppose $f \in L^p(\mathbb{R})$ and $h \in \mathbb{R}$. Define
$$\Delta_h f(x) = \frac{f(x+h) - f(x)}{h}.$$
Show that there exists $g \in L^p(\mathbb{R})$ such that
$$\lim_{h \to 0} \|\Delta_h f - g\|_p = 0$$

iff f is absolutely continuous on bounded intervals (modulo null sets) and $f' \in L^p(\mathbb{R})$. Does this remain true for $f \in L^\infty(\mathbb{R})$?

24. Suppose $f \in L^\infty(\mathbb{R})$ satisfies
25. Give an example of $f \in L^\infty(0,\infty)$ such that f' exists pointwise on $(0,\infty)$ but $f' \notin L^\infty(0,\infty)$.
26. For $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $1 < p < 2$, prove that $f * g \in L^p(\mathbb{R}^n)$ and deduce that

$$\widehat{f * g} = \hat{f}\hat{g}.$$