Advanced Course on Hardy spaces

(MA650: Assignment 2)

January- April, 2022

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) Infinite Blaschke product has only finitely many repeated factors.
   (b) Non-tangential limits and radial limits is identical for functions in $H^p(\mathbb{D}),\ 0 < p < 1$.
   (c) Can non-zero functions $f \in H^p(\mathbb{T})$, $0 < p < 1$ be zero on a set of positive measure?
   (d) If $f \in H^1(\mathbb{D})$ is outer, then necessarily $\log|f| \in L^1(\mathbb{T})$?
   (e) If $f \in L^\infty(\mathbb{T})$, then there exist two inner functions θ_1 and θ_2 and polynomial P_n such that $P_n(\bar{\theta}_1\theta_2) \to f$ uniformly.
2. Let $S_1 = \{z \in \mathbb{D} : |z-1| \le c(1-|z|)\}$. For $z = re^{i\tau}$, $|\tau| \le \pi$ and $0 < r < 1$, show that $\frac{|\tau|}{1-r}$ is uniformly bounded on S_1.
3. Show that $\mathbb{P}_+^0$ is dense in H^p for $1 \le p < \infty$ and in $H^\infty \cap C(\bar{\mathbb{D}})$.
4. Prove that H^∞ is not separable.
5. Show that $H^p \setminus H^q \neq 0$ if $q < p$.
6. Let $\xi \in \mathbb{D}$ and $1 \le p < \infty$. Define $\varphi_\xi : H^p \to \mathbb{C}$ by $\varphi_\xi(f) = f(\xi)$. Show that $\|\varphi_\xi / H^p\| = (1-|\xi|^2)^{-1/p}$.
7. The Nevanlinna class is defined by
$$\text{Nev} = N(\mathbb{D}) = \Big\{ f \in \text{Hol}(\mathbb{D}) : \sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f_r| dm < \infty \Big\},$$
   where $\log^+ t = \max(0, \log t)$ for $t > 0$ and $f_r(z) = f(rz)$.
   (i) Let $f \in N(\mathbb{D})$ and $f \neq 0$. Set $h_r(\xi) = \max(1, |f_r(\xi)|)$ for $\xi \in \mathbb{T}$ and $0 < r < 1$, and $\Phi_r = [h_r]$. Show that $\max(1, |f_r(z)|) \le |\Phi_r(z)|$ for $z \in \mathbb{D}$, and $\Phi_r(0) \le e^c$, where $c = \sup\limits_{0<r<1} \int_{\mathbb{T}} \log^+ |f_r| dm$.
   (ii) Derive from (i) that $f_r = \psi_r/\varphi_r$, where $\varphi_r = 1/\Phi_r \in H^\infty$ with $|\psi_r| \le 1$ and $\varphi_r| \le 1$ in $\mathbb{D}$ and $|\varphi_r(0)| \ge e^{-c}$ for all $0 < r < 1$. Apply the Montel compactness theorem, conclude that there exist $\varphi, \psi \in H^\infty$ such that $f = \psi/\varphi$.
   (iii) Show that $N(\mathbb{D}) = \{\psi/\varphi : \varphi, \psi \in H^\infty\} \cap \text{Hol}(\mathbb{D})$. Hence for every $f \in N(\mathbb{D})$, the non-tangential limits exist a.e., $\log|f| \in L^1$, and $f = \lambda B V_\mu[h]\,(h = |f|)$, where $V_\mu(z) = \exp\big(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\mu(\zeta)\big)$ for $|z| < 1$ for singular measure μ on $\mathbb{T}$.
   (iv) Conversely, $\lambda B V_\mu[h] \in N(\mathbb{D})$ for every λ, B, V_μ, and for all $h > 0$ with $\log h \in L^1$. Moreover, $H^p \subset l^p \cap N(\mathbb{D})$, for every $p > 0$, and $H^p = L^p \cap N_+$, where $N_+ = \{\lambda B V_\mu[h] \in N(\mathbb{D}) : \mu \ge 0\}$.
   (v) Let $f_k \in L^2(\mathbb{T})$, $k = 1, 2, \ldots, n$ and $E = \overline{\text{span}}\{z^m f_k : m \ge 0, 1 \le k \le n\}$. Show that E is simply invariant (i.e., $zE \subset E$, $zE \neq E$) if and only if
   (a) $\int_{\mathbb{T}} \log|f_k| dm > -\infty$ for all k, and
   (b) $\theta \frac{f_j}{f_k} \in N(\mathbb{D})$ for all j, k, where θ is an inner function.
8. Let $f \in N(\mathbb{D})$, $f(0) \neq 0$, and $(\lambda_n)_{n\ge1} = Z(f)$ counting multiplicity and μ satisfy $V_\mu(z) = \exp\big(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\mu(\zeta)\big)$.
   (i) Show that
$$\log|f(0)| + \sum_{n\ge1} \log \frac{1}{|\lambda|} + \mu(\mathbb{T}) = \int_{\mathbb{T}} \log|f| dm.$$

(ii) Let $f \in H^\infty$ with $|f(z)| \leq 1$ in $\mathbb{D}$, and $f(0) > 0$. Show that f is Blaschke product if and only if $\lim\limits_{r\to 1} \int\limits_{\mathbb{T}} \log|f_r| dm = 0$.

(iii) Let $f \in \mathrm{Hol}(\mathbb{D})$ and $f(0) > 0$. Show that f is a Blaschke product if and only if $\lim\limits_{r\to 1} \int\limits_{\mathbb{T}} \log|f_r| dm = 0$.

(iv) Let Let $f \in \mathrm{Hol}(\mathbb{D}_R)$, $R > 0$, and let $(\lambda_n)_{n\geq 1} = Z(f)$ counting multiplicity, let $n(s) = \mathrm{card}\{\lambda_k : |\lambda| \leq s\}$, $s \geq 0$.

(a) By assuming $f(0) \neq 0$ show that

$$\log|f(0)| + \int_0^r \frac{n(s)}{s} ds = \int_{\mathbb{T}} \log|f(r\xi) dm(\xi), \text{for all}\, r < R.$$

(b) Assume that $f(0) \neq 0$, and let $0 \leq a < R$. Show that

$$\int_a^r \frac{n(s)}{s} ds \leq \int_{\mathbb{T}} \log|f(r\xi) dm(\xi) + C$$

for $a < r < R$, where $C = C(f, a)$(constant depends on f and a).