MA642: Real Analysis -1

(Assignment 2: Metric and Normed Linear Spaces)
July - November, 2025

1. State TRUE or FALSE giving proper justification for each of the following statements.

   (a) If X is a finite metric space, does it imply that $C(X)$, the space of continuous functions on X, is a finite dimensional normed linear space?
   (b) If every countable closed set in a metric space (X, d) is complete, does it imply X is complete?
   (c) The totally boundedness property is preserved by homeomorphism.

   (d) Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n = \chi_{[0,1/n]}$ and f_n converges point-wise to f. Then the set $\{f, f_n : \; n = 1, 2, \ldots\}$ is compact in $B[0, 1]$.

   (e) Let $f_n \in C^1[0, 1]$. Then it implies that the set $\{f_n : n = 1, 2, \ldots\}$ is compact in $C[0, 1]$.

   (f) Whether $\{x = (x_1, x_2, \ldots) \in l^2 : |x_n| \leq \frac{1}{n}\}$ is totally bounded in l^2?
2. If every countable closed subset of a metric space X is complete, show that X is complete.

3. Show that a subset A of a metric space X is closed if and only if $A \cap K$ is compact for every compact set K in X.

4. Find a subset of l^∞ which is closed and bounded but not totally bounded.

5. Show that a subset A of a metric space (X, d) is totally bounded if and only if for every sequence (x_n) has a subsequence (x_{n_k}) satisfying $d(x_{n_k}, x_{n_{k+1}}) \leq 2^{-k}$.

6. Let K and F be two non-empty subsets of a metric space (X, d). If K is compact and F closed, then show that $\text{dist}(K, F) > 0$, whenever $K \cap F = \emptyset$. Does the the same conclusion holds if K is closed but not compact?

7. A function $f : (X, d) \to \mathbb{R}$ is called lower semi-continuous if for each $\alpha \in \mathbb{R}$ the set $\{x \in X : f(x) > \alpha\}$ is open in X.
   (a) Show that f is lower semi-continuous if and only if $f(x) \leq \liminf_{n\to\infty} f(x_n)$, whenever $x_n \to x$.

   (b) If X is compact metric space, prove that every lower semi-continuous function is bounded below and attains its minimum.

8. Letp $f : (X, d) \to \mathbb{R}$ be lower semi-continuous (LSC). Show that for every $x \in X$, and every sequence $x_n \to x$, implies $f(x) \leq \liminf_{n\to\infty} f(x_n)$.
9. If X is compact metric space, and let $f : X \to X$ satisfy $d(f(x), f(y)) = d(x, y)$ for $x, y \in X$. Show that f is an onto map. Is compactness of X is necessary?

10. If X is compact metric space, and let $f : X \to X$ satisfy $d(f(x), f(y)) \geq d(x, y)$ for $x, y \in X$. Show that f is an onto isometry.

11. Let X be a compact metric space, and let $f : X \to X$ be bijective and satisfy $d(f(x), f(y)) \leq d(x, y)$ for $x, y \in X$. Show that f is an isometry.

12. Let X be a compact metric space, and $\mathcal{F}$ is a subset of $(C(X)$.

(a) Prove that an equicontinuous subset $\mathcal{F}$ is pointwise bounded if and only if $\mathcal{F}$ is uniformly bounded.

(b) Prove that $\mathcal{F}$ is pointwise equicontinuous if and only if $\mathcal{F}$ uniformly equicontinuous.

13. Let X be a compact metric space, and (f_n) is a sequence in $(C(X)$.
(a) Let (f_n) be equicontinuous and pointwise convergent. Show that $f_n)$ is uniformity convergent.

(b) If (f_n) decreases pointwise to 0, show that (f_n) is equicontinuous.

(c) If (f_n) is equicontinuous, show that $\{x \in X : (f_n(x))$ converges$\}$ is a closed set in X.

14. For fixed $k > 0$ and $0 < \alpha \leq 1$, denote $\text{Lip}_k\alpha = \{f \in C[0,1] : |f(x) - f(y)| \leq k|x - y|^\alpha\}$. Show that $\{f \in \text{Lip}_k\alpha : f(0) = 0\}$ is compact subset of $C[0,1]$. Whether the set $\{f \in \text{Lip}_k\alpha : \int_0^1 f(t)dt = 1\}$ is compact?

15. Let $K(x,t)$ be a continuous function on the square $[0,1] \times [0,1]$. For $f \in C[0,1]$, define $Tf(x) = \int_0^1 f(t)K(x,t)dt$. Show that T maps bounded sets into equicontinuous sets.

16. Let $f_n \in C[0,1]$ be satisfying $\|f_n\|_\infty \leq 1$. Let $F_n(x) = \int_0^x f_n(t)dt$. Show that F_n has a convergent subsequence.

17. If $f \in B[0,1]$, show that $B_n(f)(x) \to f(x)$ at each point of continuity of f.

18. Give an example of sequence of function $f_n \in C[0,1]$, which decreases point wise to f but not uniformly.

19. For a given polynomial p and $\epsilon > 0$, show that there exits a polynomial q of rational coefficients such that $\|p - q\|_\infty < \epsilon$ on $[0,1]$.

20. Let (x_i) be a sequence in $(0,1)$ such that $\frac{1}{n}\sum\limits_{i=1}^{n} x_i^k$ is convergent for each $k = 0,1,2\ldots,$ then $\frac{1}{n}\sum\limits_{i=1}^{n} f(x_i)$ is convergent for each $f \in C[0,1]$.

21. For $f \in C[0,1]$ and $\epsilon > 0$, show that there exists a polynomial p such that $\|f - p\|_\infty < \epsilon$ and $\|f' - p'\|_\infty < \epsilon$.

22. Let $f : [1,\infty) \to \mathbb{R}$ be continuous and $\lim\limits_{x\to\infty} f(x)$ exists. For $\epsilon > 0$, show that there exists a polynomial p such that $|f(x) - p(1/x)| < \epsilon$ for all $x \geq 1$.