Assignment 2

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) $\{x \in \mathbb{R} : x^6 - 6x^4 \text{ is irrational}\}$ is a Lebesgue measurable subset of $\mathbb{R}$.
   (b) If A is a Lebesgue measurable subset of $\mathbb{R}$ and if B is a Lebesgue non-measurable subset of $\mathbb{R}$ such that $B \subset A$, then it is necessary that $m^*(A \setminus B) > 0$.
   (c) If A and B are disjoint subsets of $\mathbb{R}$ such that A is Lebesgue measurable and B is Lebesgue non-measurable, then it is possible that $m^*(A \cup B) < m^*(A) + m^*(B)$.

2. Let $A \subset [0,1]$ be Lebesgue measurable with $m(A) = 1$. If $B \subset [0,1]$, then show that $m^*(A \cap B) = m^*(B)$.

3. For $i = 1, ..., n$, let $E_i \subset (0,1)$ be Lebesgue measurable such that $\sum_{i=1}^{n} m(E_i) > n - 1$. Show that $m(\bigcap_{i=1}^{n} E_i) > 0$.

4. If $A \subset \mathbb{R}$, then show that there exists a Lebesgue measurable subset E of $\mathbb{R}$ such that $m^*(A) = m(E)$.

5. Let $A \subset \mathbb{R}$ such that $m^*(A) > 0$. Show that there exist $x, y \in A$ such that $x - y \in \mathbb{R} \setminus \mathbb{Q}$.

6. Let A and B be Lebesgue measurable subsets of $(0,1)$ such that $m(A) > \frac{1}{2}$ and $m(B) > \frac{1}{2}$. Prove that there exist $a \in A$ and $b \in B$ such that $a + b = 1$.

7. Let A be an unbounded Lebesgue measurable subset of $\mathbb{R}$ such that $m(A) < \infty$. Show that for each $\varepsilon > 0$, there exists a bounded Lebesgue measurable set B in $\mathbb{R}$ such that $B \subset A$ and $m(A \setminus B) < \varepsilon$.

8. If $A \subset \mathbb{R}$ such that $m^*(A) = 0$, then show that $m^*(\{x^2 : x \in A\}) = 0$.

9. Let $A, B \subset \mathbb{R}$ such that $A \cup B$ is Lebesgue measurable and $m(A \cup B) = m^*(A) + m^*(B) < \infty$. Show that both A and B are Lebesgue measurable.

10. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}$ and let $\{E_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint Lebesgue measurable subsets of $\mathbb{R}$ such that $A_n \subset E_n$ for each $n \in \mathbb{N}$. Show that $m^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m^*(A_n)$.

11. Let $E \subset \mathbb{R}$ and let $\alpha \in \mathbb{R}$. If $\alpha E = \{\alpha x : x \in E\}$, then show that $m^*(\alpha E) = |\alpha| m^*(E)$. Also, show that if E is Lebesgue measurable, then αE is Lebesgue measurable.

12. If E is a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) < +\infty$ and if $f(x) = m(E \cap (-\infty, x])$ for all $x \in \mathbb{R}$, then show that $f : \mathbb{R} \to \mathbb{R}$ is continuous.

13. Let $E \subset \mathbb{R}$ and $m^*(E) > 0$. Then for each $0 < \alpha < 1$, there exists an open interval I such that $m^*(E \cap I) \geq \alpha m(I)$.

14. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $m(E) < \infty$. Then there exist a sequence of compact set (K_n) contained in E and a set N Lebesgue measure zero such that $E = F \cup N$, where $F = \cup_{n=1}^{\infty} K_n$.