Assignment 2

1. State TRUE or FALSE giving proper justification for each of the following statements.

   (a) Let $\mathbb{P}(\mathbb{R})$ be the space of all polynomials with coefficients in $\mathbb{R}$. Does there exist a norm $\|\cdot\|$ on $\mathbb{P}(\mathbb{R})$ such that $(\mathbb{P}(\mathbb{R}), \|.\|)$ is a Banach space?

   (b) Let $C_c(\mathbb{R})$ be the class of all compactly supported continuous functions on $\mathbb{R}$. Does the linear functional given by $T(f) = \int_{-\infty}^{\infty} f(t)dt$ is continuous in $(C_c(\mathbb{R}), \|\cdot\|_\infty)$? Whether T is continuous in $(C_c(\mathbb{R}), \|\cdot\|_1)$?

2. Show that every infinite-dimensional separable normed linear space contains a countable linearly independent dense subset.

3. Let X be a separable Banach space. Prove that there exists a closed subspace M of l^1 such that X is isomorphic to l^1/M.

4. Prove that an infinite dimensional Banach space X can not be expressed as the countable union of compact subsets of X.

5. Let $f \in C^\infty(\mathbb{R})$ be such that for each $t \in \mathbb{R}$, there exists $n_t \in \mathbb{N}$ satisfying $f^{(n_t)}(t) = 0$. Show that there some interval (a, b) and polynomial $p(x)$ such that $f(t) = p(t)$ for all $t \in (a, b)$.

6. Let X, Y be normed linear spaces and let $T : X \to Y$ and $S : X \to Y$ be linear maps. Let $x_0 \in X$ and $r > 0$ such that $Tx = Sx$ for all $x \in B_r(x_0)$. Show that $T = S$.

7. Let X, Y be normed linear spaces and let $T : X \to Y$ be linear such that $\{Tx : x \in X, \|x\| < 1\}$ is an open set in Y. If G is an open set in X, then show that $T(G)$ is an open set in Y.

8. Let X, Y be normed linear spaces and let $T : X \to Y$ be a linear map. Show that the following statements are equivalent.
   (a) T is bounded.
   (b) For every Cauchy sequence (x_n) in X, (Tx_n) is a Cauchy sequence in Y.
   (c) For every nonempty bounded open subset G of X, $T(G)$ is a bounded subset of Y.

9. Let X, Y be normed linear spaces and let $T : X \to Y$ be a linear map. If $T(G)$ is a bounded subset of Y for some nonempty open subset G of X, then show that T is continuous.

10. Let X, Y be normed linear spaces and let $T : X \to Y$ be a linear map such that $\{x \in X : \|Tx\| < 1\}^0 \neq \emptyset$. Show that T is continuous.

11. Let X, Y be normed linear spaces and let $T : X \to Y$ be linear. If $T(K)$ is a bounded set in Y for every compact set K in X, then show that T is continuous.

12. Let $T_n : l^1 \to l^1$ be a sequence of linear transformations such that for each $x = (x_n) \in l^1$, $T_n(x_1, x_2, \ldots) = (x_{n+1}, x_{n+2}, \ldots)$. Show that $\|T_n(x)\| \to 0$ but $\|T_n\| = 1$.

13. For $f \in C^1[0, 1]$, define its norm by $\|f\| = \max\{\|f\|_\infty, \|f'\|_\infty\}$. Show that the linear map $T : (C^1[0, 1], \|.\|) \to (C[0, 1], \|.\|_\infty)$ given by $T(f) = f'$ is continuous and $\|T\| = 1$.

14. Let X, Y be normed linear spaces and let $T : X \to Y$ be a linear map. If for every absolutely convergent series $\sum\limits_{n=1}^{\infty} x_n$ in X, $\sum\limits_{n=1}^{\infty} Tx_n$ is a convergent series in Y, then show that T is bounded.

15. For $1 \leq p \leq \infty$, define a linear map on $l^p(\mathbb{N})$ by $T(x) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$. Find $\|T\|$.

16. Let $X = (C[0,1], \|\cdot\|_\infty)$. For $f \in C[0,1]$ define $K(f)(t) = \int\limits_0^t f(s)ds$. Show that

   (a) K is one one but not onto.

   (b) For each $n \in \mathbb{N}$, the power of operator K satisfies $\|K^n\| = \frac{1}{n!}$.

   (c) Operator $T = I + K$ is invertible.

17. Let $X = (C[0,\pi], \|\cdot\|_\infty)$. For $f \in C[0,\pi]$ define $T(f)(x) = \int_0^x \sin(x+y)f(y)dy$. Find $\|T\|$.

18. Let X and Y be two Banach spaces and $T : X \to Y$ is a continuous linear bijection. Then show that $\inf\{\|x\| : x \in X,\ \|Tx\| = 1\} \leq \|T^{-1}\|$. Does equality hold?

19. Let $\phi \in L^\infty(\mathbb{R})$. For $1 \leq p < \infty$, define an operator on $L^p(\mathbb{R}^n)$ by $M_\phi(f) = \phi f$. Show that $\|M_\phi\| = \|\phi\|_\infty$. Whether the conclusion is true for $p = \infty$?

20. Let M be a closed subspace of a normed linear space X. Show that the projection $\pi :\ X \to X/M$ defined by $\pi(x) = x + M$ is a continuous linear surjective open map with $\|\pi\| < 1$. If $M \subsetneq X$ then $\|\pi\| = 1$.

21. Let $X = \{f \in C[0,1] :\ f(0) = 0\}$ and $M = \left\{f \in X :\ \int_0^1 f(t)dt = 0\right\}$. Show that

   (a) M is an infinite dimensional closed subspace of X.

   (b) There does not exists any $f \in X$ with $\|f\|_\infty = 1$ such that $\text{dist}(f, M) = 1$.

22. Let X and Y be two Banach spaces and $T \in B(X,Y)$. Show that followings are equivalent:

   (a) T is injective and has closed range.

   (b) There is $k \geq 0$ such that $\|x\| \leq k\|T(x)\|$ for all $x \in X$.

23. Let X and Y be two normed linear spaces and $T : X \to Y$ such that $\dim T(X) < \infty$ and $\ker T$ is closed. Then show that T is bounded.

24. Suppose X can be made Banach space with respect to norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If there exists $m > 0$ such that $\|x\|_1 \leq m\|x\|_2$ for all $x \in X$. Then both norms are equivalent.

25. Let X be a Banach space and Y be a normed linear space. Suppose $T_n \in B(X,Y)$ such that $\lim\limits_{n\to\infty} T_n(x)$ exists for each $x \in X$. Write $T(x) = \lim\limits_{n\to\infty} T_n(x)$. Show that T is bounded.

26. Let (a_n) be a sequence of real numbers such that for each $x = (x_n) \in l^2$, the sequence $(a_n x_n) \in l^2(\mathbb{N})$. Define an operator T on l^2 by $T(x_1, x_2, \ldots) = (a_1x_1, a_2x_2, \ldots)$. Show that T is bounded.

27. Let X and Y be two normed linear spaces and $T_n, T \in B(X,Y)$ such that $T_n \to T$. Suppose $x_n \to x$. Then show that $T_n x_n \to Tx$ in Y.

28. Let X be a Banach space and let $T \in B(X)$ with $\|T\| < 1$. Then show that $I - T$ is invertible and $(I-T)^{-1} = \sum\limits_{n=0}^{\infty} T^n \in B(X)$.

29. Let (T_n) be a sequence of bounded linear operator on a Banach space X such that $\|T_n - T\| \to 0$. If T_n^{-1} exists, $\forall\ n \in \mathbb{N}$ and $\|T_n^{-1}\| < 1$, then prove that $T^{-1} \in B(X)$.

30. Let X and Y be two normed linear spaces and $T \in B(X,Y)$ sends each open subset in X to an open subset in Y. Prove that T is onto.

31. Let $T : (C^1[0,1], \|\ .\ \|_\infty) \to (C[0,1], \|\ .\ \|_\infty)$ be a linear transformation that defined by $Tf(s) = f'(s) + \int_0^s f(t)dt$. Show that the graph of T is closed but T is not continuous.