Assignment 2: Metric and Normed Linear Spaces.

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) There exists a metric space having exactly 36 open sets.
   (b) It is impossible to define a metric d on $\mathbb{R}$ such that only finitely many subsets of $\mathbb{R}$ are open in $(\mathbb{R}, d)$.
   (c) If A and B are open (closed) subsets of a normed vector space X, then $A+B = \{a+b : a \in A, b \in B\}$ is open (closed) in X.
   (d) If A and B are closed subsets of $[0,\infty)$ (with the usual metric), then $A+B$ is closed in $[0,\infty)$.
   (e) It is possible to define a metric d on $\mathbb{R}$ such that the sequence $(1,0,1,0,...)$ converges in $(\mathbb{R}, d)$.
   (f) It is possible to define a metric d on $\mathbb{R}^2$ such that $((\frac{1}{n}, \frac{n}{n+1}))$ is not a Cauchy sequence in $(\mathbb{R}^2, d)$.
   (g) It is possible to define a metric d on $\mathbb{R}^2$ such that in $(\mathbb{R}^2, d)$, the sequence $((\frac{1}{n}, 0))$ converges but the sequence $((\frac{1}{n}, \frac{1}{n}))$ does not converge.
   (h) If (x_n) is a sequence in a complete normed vector space X such that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, then (x_n) must converge in X.
   (i) If (f_n) is a sequence in $C[0,1]$ such that $|f_{n+1}(x) - f_n(x)| \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0,1]$, then there must exist $f \in C[0,1]$ such that $\int_0^1 |f_n(x) - f(x)|\, dx \to 0$ as $n \to \infty$.
   (j) If (x_n) is a Cauchy sequence in a normed vector space, then $\lim\limits_{n\to\infty} \|x_n\|$ must exist.
   (k) $\{f \in C[0,1] : \|f\|_1 \leq 1\}$ is a bounded subset of the normed vector space $(C[0,1], \|\cdot\|_\infty)$.

2. Examine whether (X, d) is a metric space, where
   (a) $X = \mathbb{R}$ and $d(x,y) = \frac{|x-y|}{1+|xy|}$ for all $x, y \in \mathbb{R}$.
   (b) $X = \mathbb{R}$ and $d(x,y) = \min\{\sqrt{|x-y|}, |x-y|^2\}$ for all $x, y \in \mathbb{R}$.
   (c) $X = \mathbb{R}$ and $d(x,y) = |x-y|^p$ for all $x, y \in \mathbb{R}$ $(0 < p < 1)$.
   (d) $X = \mathbb{R}$ and for all $x, y \in \mathbb{R}$, $d(x,y) = \begin{cases} 1 + |x-y| & \text{if exactly one of } x \text{ and } y \text{ is positive,} \\ |x-y| & \text{otherwise.} \end{cases}$
   (e) $X = \mathbb{R}^2$ and $d(x,y) = (|x_1 - y_1| + |x_2 - y_2|^{\frac{1}{2}})^{\frac{1}{2}}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.
   (f) $X = \mathbb{R}^n$ and $d(x,y) = [(x_1 - y_1)^2 + \frac{1}{2}(x_2 - y_2)^2 + \cdots + \frac{1}{n}(x_n - y_n)^2]^{\frac{1}{2}}$ for all $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$.
   (g) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z,w) = \begin{cases} \min\{|z| + |w|, |z-1| + |w-1| & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$
   (h) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z,w) = \begin{cases} |z-w| & \text{if } \frac{z}{|z|} = \frac{w}{|w|}, \\ |z| + |w| & \text{otherwise.} \end{cases}$
   (i) $X = \mathbb{C}$ and $d(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z, w \in \mathbb{C}$.
   (j) $X =$ The class of all finite subsets of a nonempty set and $d(A,B) =$ The number of elements of the set $A \triangle B$ (the symmetric difference of A and B).
   (k) $X = C[0,1]$ and $d(f,g) = (\int_0^1 |f(t) - g(t)|^2\, dt)^{\frac{1}{2}}$ for all $f, g \in C[0,1]$.

3. Examine whether $\|\cdot\|$ is a norm on $\mathbb{R}^2$, where for each $(x,y) \in \mathbb{R}^2$,
   (a) $\|(x,y)\| = (\sqrt{|x|} + \sqrt{|y|})^2$.
   (b) $\|(x,y)\| = \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}$.
   (c) $\|(x,y)\| = \begin{cases} \sqrt{x^2 + y^2} & \text{if } xy \geq 0, \\ \max\{|x|, |y|\} & \text{if } xy < 0. \end{cases}$

4. Let $\|f\| = \min\{\|f\|_\infty, 2\|f\|_1\}$ for all $f \in C[0,1]$. Prove that $\|\cdot\|$ is not a norm on $C[0,1]$.

5. If $\mathbf{x} \in \mathbb{R}^n$, then show that $\lim_{p\to\infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$.

6. If $1 \le p < q \le \infty$, then show that $\|x\|_q \le \|x\|_p$ for all $x \in \ell^p$.

7. Let d be a metric on a real vector space X satisfying the following two conditions:
   (i) $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in X$,
   (ii) $d(\alpha x, \alpha y) = |\alpha| d(x,y)$ for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$.
   Show that there exists a norm $\|\cdot\|$ on X such that $d(x,y) = \|x-y\|$ for all $x, y \in X$.

8. Let $\mathbb{R}^\infty$ be the real vector space of all sequences in $\mathbb{R}$, where addition and scalar multiplication are defined componentwise. Let $d((x_n),(y_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1+|x_n-y_n|}$ for all $(x_n),(y_n) \in \mathbb{R}^\infty$. Show that d is a metric on $\mathbb{R}^\infty$ but that no norm on $\mathbb{R}^\infty$ induces d.

9. Let $(X, \|\cdot\|)$ be a nonzero normed vector space. Consider the metrics d_1, d_2 and d_3 on X:
$$d_1(x,y) := \min\{1, \|x-y\|\},$$
$$d_2(x,y) := \frac{\|x-y\|}{1+\|x-y\|},$$
$$d_3(x,y) := \begin{cases} 1+\|x-y\| & \text{if } x \ne y, \\ 0 & \text{if } x = y, \end{cases}$$
   for all $x, y \in X$. Prove that none of d_1, d_2 and d_3 is induced by any norm on X.

10. Let X be a normed vector space containing more than one point, let $x, y \in X$ and let $\varepsilon, \delta > 0$. If $B_\varepsilon[x] = B_\delta[y]$, show that $x = y$ and $\varepsilon = \delta$. Does the result remain true if X is assumed to be a metric space? Justify.

11. Examine whether the following sets are open/closed in $\mathbb{R}^2$ (with the usual metric).
   (a) $\{(x,y) \in \mathbb{R}^2 : xy > 0\}$
   (b) $\{(x,x) : x \in \mathbb{R}\}$
   (c) $(0,1) \times \{0\}$
   (d) $\{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$
   (e) $\{(x,y) \in \mathbb{R}^2 : x + y < 1\}$
   (f) $\{(x,y) \in \mathbb{R}^2 : y \in \mathbb{Z}\}$

12. Let $A = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is a closed/an open subset of $\mathbb{R}^3$ with respect to the usual metric on $\mathbb{R}^3$.

13. Examine whether a finite subset of a metric space is open/closed.

14. For all $x, y \in \mathbb{R}$, let $d_1(x,y) = |x-y|$, $d_2(x,y) = \min\{1, |x-y|\}$ and $d_3(x,y) = \frac{|x-y|}{1+|x-y|}$. If G is an open set in any one of the three metric spaces $(\mathbb{R}, d_i)$ $(i = 1,2,3)$, then show that G is also open in the other two metric spaces.

15. Let X be a nonzero normed vector space. Show that $\{x \in X : \|x\| < 1\}$ is not closed in X and $\{x \in X : \|x\| \le 1\}$ is not open in X.

16. Let X be a normed vector space and let $Y(\neq X)$ be a subspace of X. Show that Y is not open in X.

17. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X,d). Show that the sequence $(d(x_n,y_n))$ is convergent.

18. Let (x_n) be a sequence in a complete metric space (X,d) such that $\sum\limits_{n=1}^{\infty} d(x_n,x_{n+1}) < \infty$. Show that (x_n) converges in (X,d).

19. Let (x_n) be a sequence in a metric space X such that each of the subsequences (x_{2n}), (x_{2n-1}) and (x_{3n}) converges in X. Show that (x_n) converges in X.

20. Show that the following are incomplete metric spaces.
(a) $(\mathbb{N},d)$, where $d(m,n) = |\frac{1}{m} - \frac{1}{n}|$ for all $m,n \in \mathbb{N}$
(b) $((0,\infty),d)$, where $d(x,y) = |\frac{1}{x} - \frac{1}{y}|$ for all $x,y \in (0,\infty)$
(c) $(\mathbb{R},d)$, where $d(x,y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$ for all $x,y \in \mathbb{R}$
(d) $(\mathbb{R},d)$, where $d(x,y) = |e^x - e^y|$ for all $x,y \in \mathbb{R}$

21. Examine whether the following metric spaces are complete.
(a) $([0,1),d)$, where $d(x,y) = |\frac{x}{1-x} - \frac{y}{1-y}|$ for all $x,y \in [0,1)$
(b) $((-1,1),d)$, where $d(x,y) = |\tan\frac{\pi x}{2} - \tan\frac{\pi y}{2}|$ for all $x,y \in (-1,1)$

22. For $X(\neq \emptyset) \subset \mathbb{R}$, let $d(x,y) = \frac{|x-y|}{1+|x-y|}$ for all $x,y \in X$. Examine the completeness of the metric space (X,d), where X is
(a) $[0,1] \cap \mathbb{Q}$.
(b) $[-1,0] \cup [1,\infty)$.
(c) $\{n^2 : n \in \mathbb{N}\}$.

23. Examine whether the sequence (f_n) is convergent in $(C[0,1],d_\infty)$, where for all $n \in \mathbb{N}$ and for all $t \in [0,1]$,
(a) $f_n(t) = \frac{nt^2}{1+nt}$.
(b) $f_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$.
(c) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{1}{nt} & \text{if } \frac{1}{n} < t \le 1. \end{cases}$
(d) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{n}{n-1}(1-t) & \text{if } \frac{1}{n} < t \le 1. \end{cases}$

24. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let there exist $\alpha > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{y})\| \ge \alpha\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$. Prove that f is one-one, onto and that $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

25. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction and let $g(\mathbf{x}) = \mathbf{x} - f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that $g : \mathbb{R}^n \to \mathbb{R}^n$ is one-one and onto. Also, show that both g and $g^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.