

# MA211(M): Real Analysis

(Assignment 2: Metric and Normed Linear Spaces)

July - November, 2022

1. State TRUE or FALSE giving proper justification for each of the following statements.
  - (a) There exists a metric space having exactly 36 open sets.
  - (b) It is impossible to define a metric  $d$  on  $\mathbb{R}$  such that only finitely many subsets of  $\mathbb{R}$  are open in  $(\mathbb{R}, d)$ .
  - (c) If  $A$  and  $B$  are open (closed) subsets of a normed vector space  $X$ , then  $A + B = \{a + b : a \in A, b \in B\}$  is open (closed) in  $X$ .
  - (d) If  $A$  and  $B$  are closed subsets of  $[0, \infty)$  (with the usual metric), then  $A + B$  is closed in  $[0, \infty)$ .
  - (e) It is possible to define a metric  $d$  on  $\mathbb{R}$  such that the sequence  $(1, 0, 1, 0, \dots)$  converges in  $(\mathbb{R}, d)$ .
  - (f) It is possible to define a metric  $d$  on  $\mathbb{R}^2$  such that  $((\frac{1}{n}, \frac{n}{n+1}))$  is not a Cauchy sequence in  $(\mathbb{R}^2, d)$ .
  - (g) It is possible to define a metric  $d$  on  $\mathbb{R}^2$  such that in  $(\mathbb{R}^2, d)$ , the sequence  $((\frac{1}{n}, 0))$  converges but the sequence  $((\frac{1}{n}, \frac{1}{n}))$  does not converge.
  - (h) There exist two non-empty disjoint sets  $A$  and  $B$  in  $\mathbb{R}$  such that  $\inf\{|x - y| : x \in A \text{ and } y \in B\} = 0$ .
  - (i) If  $(x_n)$  is a sequence in a complete normed vector space  $X$  such that  $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ , then  $(x_n)$  must converge in  $X$ .
  - (j) If  $(f_n)$  is a sequence in  $C[0, 1]$  such that  $|f_{n+1}(x) - f_n(x)| \leq \frac{1}{n^2}$  for all  $n \in \mathbb{N}$  and for all  $x \in [0, 1]$ , then there must exist  $f \in C[0, 1]$  such that  $\int_0^1 |f_n(x) - f(x)| dx \rightarrow 0$  as  $n \rightarrow \infty$ .
  - (k) If  $(x_n)$  is a Cauchy sequence in a normed vector space, then  $\lim_{n \rightarrow \infty} \|x_n\|$  must exist.
  - (l)  $\{f \in C[0, 1] : \|f\|_1 \leq 1\}$  is a bounded subset of the normed vector space  $(C[0, 1], \|\cdot\|_\infty)$ .
  - (m) There exist a set  $A \subset (\mathbb{R}, u)$  such that  $\delta(A \cup \{0\}) = 0$  but  $\delta(\bar{A}) = 1$ , where  $\delta$  stands for diameter.
  - (n) For  $x, y \in l^\infty$ ,  $d(x, y) = \min\{1, \limsup |x_n - y_n|\}$  define a metric on  $l^\infty$ .
  - (o) The sequence  $f_n(t) = e^{-n^2 \sin \pi t}$  converge uniformly to 0 on  $(0, 1)$ .
2. Examine whether  $(X, d)$  is a metric space, where
  - (a)  $X = \mathbb{R}$  and  $d(x, y) = \frac{|x-y|}{1+|xy|}$  for all  $x, y \in \mathbb{R}$ .
  - (b)  $X = \mathbb{R}$  and  $d(x, y) = \min\{\sqrt{|x-y|}, |x-y|^2\}$  for all  $x, y \in \mathbb{R}$ .
  - (c)  $X = \mathbb{R}$  and  $d(x, y) = |x-y|^p$  for all  $x, y \in \mathbb{R}$  ( $0 < p < 1$ ).
  - (d)  $X = \mathbb{R}$  and for all  $x, y \in \mathbb{R}$ ,  $d(x, y) = \begin{cases} 1 + |x-y| & \text{if exactly one of } x \text{ and } y \text{ is positive,} \\ |x-y| & \text{otherwise.} \end{cases}$
  - (e)  $X = \mathbb{R}^2$  and  $d(x, y) = (|x_1 - y_1| + |x_2 - y_2|^{\frac{1}{2}})^{\frac{1}{2}}$  for all  $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ .
  - (f)  $X = \mathbb{R}^n$  and  $d(x, y) = [(x_1 - y_1)^2 + \frac{1}{2}(x_2 - y_2)^2 + \dots + \frac{1}{n}(x_n - y_n)^2]^{\frac{1}{2}}$  for all  $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$ .
  - (g)  $X = \mathbb{C}$  and for all  $z, w \in \mathbb{C}$ ,  $d(z, w) = \begin{cases} \min\{|z| + |w|, |z-1| + |w-1|\} & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$
  - (h)  $X = \mathbb{C}$  and for all  $z, w \in \mathbb{C}$ ,  $d(z, w) = \begin{cases} |z-w| & \text{if } \frac{z}{|z|} = \frac{w}{|w|}, \\ |z| + |w| & \text{otherwise.} \end{cases}$
  - (i)  $X = \mathbb{C}$  and  $d(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$  for all  $z, w \in \mathbb{C}$ .
  - (j)  $X =$  The class of all finite subsets of a nonempty set and  $d(A, B) =$  The number of elements of the set  $A \Delta B$  (the symmetric difference of  $A$  and  $B$ ).

(k)  $X = C[0, 1]$  and  $d(f, g) = \left(\int_0^1 |f(t) - g(t)|^2 dt\right)^{\frac{1}{2}}$  for all  $f, g \in C[0, 1]$ .

3. Examine whether  $\|\cdot\|$  is a norm on  $\mathbb{R}^2$ , where for each  $(x, y) \in \mathbb{R}^2$ ,

(a)  $\|(x, y)\| = (\sqrt{|x|} + \sqrt{|y|})^2$ .

(b)  $\|(x, y)\| = \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}$ .

(c)  $\|(x, y)\| = \begin{cases} \sqrt{x^2 + y^2} & \text{if } xy \geq 0, \\ \max\{|x|, |y|\} & \text{if } xy < 0. \end{cases}$

4. Let  $\|f\| = \min\{\|f\|_\infty, 2\|f\|_1\}$  for all  $f \in C[0, 1]$ . Prove that  $\|\cdot\|$  is not a norm on  $C[0, 1]$ .

5. If  $\mathbf{x} \in \mathbb{R}^n$ , then show that  $\lim_{p \rightarrow \infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$ .

6. If  $1 \leq p < q \leq \infty$ , then show that  $\|x\|_q \leq \|x\|_p$  for all  $x \in \ell^p$ .

7. Let  $d$  be a metric on a real vector space  $X$  satisfying the following two conditions:

(i)  $d(x + z, y + z) = d(x, y)$  for all  $x, y, z \in X$ ,

(ii)  $d(\alpha x, \alpha y) = |\alpha|d(x, y)$  for all  $x, y \in X$  and for all  $\alpha \in \mathbb{R}$ .

Show that there exists a norm  $\|\cdot\|$  on  $X$  such that  $d(x, y) = \|x - y\|$  for all  $x, y \in X$ .

8. Let  $\mathbb{R}^\infty$  be the real vector space of all sequences in  $\mathbb{R}$ , where addition and scalar multiplication are defined componentwise. Let  $d((x_n), (y_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}$  for all  $(x_n), (y_n) \in \mathbb{R}^\infty$ . Show that  $d$  is a metric on  $\mathbb{R}^\infty$  but that no norm on  $\mathbb{R}^\infty$  induces  $d$ .

9. Let  $(X, \|\cdot\|)$  be a nonzero normed vector space. Consider the metrics  $d_1, d_2$  and  $d_3$  on  $X$ :

$$d_1(x, y) := \min\{1, \|x - y\|\},$$

$$d_2(x, y) := \frac{\|x - y\|}{1 + \|x - y\|},$$

$$d_3(x, y) := \begin{cases} 1 + \|x - y\| & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for all  $x, y \in X$ . Prove that none of  $d_1, d_2$  and  $d_3$  is induced by any norm on  $X$ .

10. Let  $X$  be a normed vector space containing more than one point, let  $x, y \in X$  and let  $\varepsilon, \delta > 0$ . If  $B_\varepsilon[x] = B_\delta[y]$ , show that  $x = y$  and  $\varepsilon = \delta$ . Does the result remain true if  $X$  is assumed to be a metric space? Justify.

11. Examine whether the following sets are open/closed in  $\mathbb{R}^2$  (with the usual metric).

(a)  $\{(x, y) \in \mathbb{R}^2 : xy > 0\}$

(b)  $\{(x, x) : x \in \mathbb{R}\}$

(c)  $(0, 1) \times \{0\}$

(d)  $\{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$

(e)  $\{(x, y) \in \mathbb{R}^2 : x + y < 1\}$

(f)  $\{(x, y) \in \mathbb{R}^2 : y \in \mathbb{Z}\}$

12. Let  $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$  and  $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ . Examine whether  $A \cap B$  is a closed/an open subset of  $\mathbb{R}^3$  with respect to the usual metric on  $\mathbb{R}^3$ .

13. Examine whether a finite subset of a metric space is open/closed.
14. For all  $x, y \in \mathbb{R}$ , let  $d_1(x, y) = |x - y|$ ,  $d_2(x, y) = \min\{1, |x - y|\}$  and  $d_3(x, y) = \frac{|x-y|}{1+|x-y|}$ . If  $G$  is an open set in any one of the three metric spaces  $(\mathbb{R}, d_i)$  ( $i = 1, 2, 3$ ), then show that  $G$  is also open in the other two metric spaces.
15. Let  $X$  be a nonzero normed vector space. Show that  $\{x \in X : \|x\| < 1\}$  is not closed in  $X$  and  $\{x \in X : \|x\| \leq 1\}$  is not open in  $X$ .
16. Show that  $A = \{f \in C[0, 1] : \|f\|_1 < 1\}$  is an unbounded subset of the normed linear space  $(C[0, 1], \|\cdot\|_\infty)$ .
17. Let  $X$  be a normed vector space and let  $Y (\neq X)$  be a subspace of  $X$ . Show that  $Y$  is not open in  $X$ .
18. Let  $F_n$  be a sequence of closed sets in  $\mathbb{R}$  such that  $F_n \subset (n, n + 1]$  and  $F_n \cap F_m = \emptyset$ , whenever  $m \neq n$ . Show that  $F = \bigcup_{n=1}^{\infty} F_n$  is a closed set in  $\mathbb{R}$ .
19. Let  $f : (X, d) \rightarrow \mathbb{R}$  be a continuous function. Show that  $\{x \in X : f(x) \neq 0\}$  is an open set in the metric space  $(X, d)$ .
20. Let  $(x_n)$  and  $(y_n)$  be Cauchy sequences in a metric space  $(X, d)$ . Show that the sequence  $(d(x_n, y_n))$  is convergent.
21. Let  $(x_n)$  be a sequence in a complete metric space  $(X, d)$  such that  $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ . Show that  $(x_n)$  converges in  $(X, d)$ .
22. Let  $(x_n)$  be a sequence in a metric space  $X$  such that each of the subsequences  $(x_{2n})$ ,  $(x_{2n-1})$  and  $(x_{3n})$  converges in  $X$ . Show that  $(x_n)$  converges in  $X$ .
23. Show that the following are incomplete metric spaces.
- $(\mathbb{N}, d)$ , where  $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$  for all  $m, n \in \mathbb{N}$
  - $((0, \infty), d)$ , where  $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$  for all  $x, y \in (0, \infty)$
  - $(\mathbb{R}, d)$ , where  $d(x, y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$  for all  $x, y \in \mathbb{R}$
  - $(\mathbb{R}, d)$ , where  $d(x, y) = |e^x - e^y|$  for all  $x, y \in \mathbb{R}$
24. Examine whether the following metric spaces are complete.
- $([0, 1), d)$ , where  $d(x, y) = |\frac{x}{1-x} - \frac{y}{1-y}|$  for all  $x, y \in [0, 1)$
  - $((-1, 1), d)$ , where  $d(x, y) = |\tan \frac{\pi x}{2} - \tan \frac{\pi y}{2}|$  for all  $x, y \in (-1, 1)$
25. For  $X (\neq \emptyset) \subset \mathbb{R}$ , let  $d(x, y) = \frac{|x-y|}{1+|x-y|}$  for all  $x, y \in X$ . Examine the completeness of the metric space  $(X, d)$ , where  $X$  is
- $[0, 1] \cap \mathbb{Q}$ .
  - $[-1, 0] \cup [1, \infty)$ .
  - $\{n^2 : n \in \mathbb{N}\}$ .
26. Examine whether the sequence  $(f_n)$  is convergent in  $(C[0, 1], d_\infty)$ , where for all  $n \in \mathbb{N}$  and for all  $t \in [0, 1]$ ,
- $f_n(t) = \frac{nt^2}{1+nt}$ .

$$(b) f_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}.$$

$$(c) f_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n}, \\ \frac{1}{nt} & \text{if } \frac{1}{n} < t \leq 1. \end{cases}$$

$$(d) f_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n}, \\ \frac{n}{n-1}(1-t) & \text{if } \frac{1}{n} < t \leq 1. \end{cases}$$

27. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be continuous and let there exist  $\alpha > 0$  such that  $\|f(\mathbf{x}) - f(\mathbf{y})\| \geq \alpha\|\mathbf{x} - \mathbf{y}\|$  for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ . Show that  $f(\mathbb{R}^n)$  is complete.
28. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a contraction and let  $g(\mathbf{x}) = \mathbf{x} - f(\mathbf{x})$  for all  $\mathbf{x} \in \mathbb{R}^n$ . Show that  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is one-one and onto. Also, show that both  $g$  and  $g^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous.