MA15010H: Multi-variable Calculus

(Assignment 1 Hint/model solutions: Sequential continuity and vector differentiability)
September - November, 2025

Question 0.1. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is (a) an open set (b) a closed set in $\mathbb{R}^3$.

solution 0.2. *We have $(0, 0, 0) \in A \cap B$. If possible, let $(0, 0, 0) \in (A \cap B)^0$. Then there exists $r > 0$ such that $B_r((0, 0, 0)) \subseteq A \cap B$. Since $(0, 0, \frac{r}{2}) \in B_r((0, 0, 0))$ but $(0, 0, \frac{r}{2}) \notin A \cap B$, we get a contradiction. Hence $(0, 0, 0) \notin (A \cap B)^0$. Therefore $A \cap B$ is not an open set in $\mathbb{R}^3$.*

Again, since $(1 - \frac{1}{n}, 0, 0) \in A \cap B$ for all $n \in \mathbb{N}$ and since $(1 - \frac{1}{n}, 0, 0) \to (1, 0, 0) \notin A \cap B$, $A \cap B$ is not a closed set in $\mathbb{R}^3$.

Question 0.3. Show that $\{x \in \mathbb{R}^m : 1 < \|x\| \leq 2\}$ is neither an open set nor a closed set in $\mathbb{R}^m$.

solution 0.4. *Let $S = \{x \in \mathbb{R}^m : 1 < \|x\| \leq 2\}$. Since $\|(2 + \frac{1}{n})e_1\| = 2 + \frac{1}{n} > 2$ for all $n \in \mathbb{N}$, $(2 + \frac{1}{n})e_1 \in \mathbb{R}^m \setminus S$ for all $n \in \mathbb{N}$. Also, $(2 + \frac{1}{n})e_1 \to 2e_1 \notin \mathbb{R}^m \setminus S$, since $\|2e_1\| = 2$. Hence $\mathbb{R}^m \setminus S$ is not a closed set in $\mathbb{R}^m$ and consequently S is not an open set in $\mathbb{R}^m$.*

Again, since $\|(1 + \frac{1}{n})e_1\| = 1 + \frac{1}{n} \in (1, 2]$ for all $n \in \mathbb{N}$, $(1 + \frac{1}{n})e_1 \in S$ for all $n \in \mathbb{N}$. Also, $(1 + \frac{1}{n})e_1 \to e_1 \notin S$, since $\|e_1\| = 1$. Hence S is not a closed set in $\mathbb{R}^m$.

Question 0.5. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and if S is a bounded subset of $\mathbb{R}^2$, then $f(S)$ must be a bounded subset of $\mathbb{R}$.

solution 0.6. *Since S is a bounded subset of $\mathbb{R}^2$, there exists $r > 0$ such that $S \subseteq B_r[0]$. Now, since $B_r[0]$ is a closed and bounded set in $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $f(B_r[0])$ is a bounded set in $\mathbb{R}$. Hence there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in B_r[0]$. So, in particular, $|f(x)| \leq M$ for all $x \in S$. Hence $f(S)$ is a bounded subset of $\mathbb{R}$. Therefore the given statement is TRUE.*

Question 0.7. Let S be a nonempty subset of $\mathbb{R}^m$ such that every continuous function $f : S \to \mathbb{R}$ is bounded. Show that S is a closed and bounded set in $\mathbb{R}^m$.

solution 0.8. *If possible, let S be not closed in $\mathbb{R}^m$. Then there exists $x_0 \in \mathbb{R}^m \setminus S$ and a sequence (x_n) in S such that $x_n \to x_0$. The function $f : S \to \mathbb{R}$, defined by $f(x) = \frac{1}{\|x - x_0\|}$ for all $x \in S$, is continuous but not bounded (since $\|x_n - x_0\| \to 0$ and so $f(x_n) \to \infty$), which contradicts the hypothesis. Hence S must be a closed set in $\mathbb{R}^m$.*

Again, if possible, let S be not bounded in $\mathbb{R}^m$. Then the function $g : S \to \mathbb{R}$, defined by $g(x) = \|x\|$ for all $x \in S$, is continuous but not bounded, which contradicts the hypothesis. Hence S must be bounded in $\mathbb{R}^m$.

Question 0.9. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ and let $f : S \to \mathbb{R}$ be continuous. Show that there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ such that $f(S) = [\alpha, \beta]$.

solution 0.10. *We know that* $S = B_1[0]$ *is a closed and bounded set in* $\mathbb{R}^3$. *Since* $f : S \to \mathbb{R}$ *is continuous, there exist* $x_0, y_0 \in S$ *such that*

$$f(x_0) \le f(x) \le f(y_0) \quad \textit{for all } x \in S.$$

Taking $\alpha = f(x_0)$ *and* $\beta = f(y_0)$, *we find that* $\alpha, \beta \in \mathbb{R}, \alpha \le \beta$, *and*

$$f(S) \subseteq [\alpha, \beta].$$

Again, if $t \in [0,1]$, *then* $(1-t)x_0 + ty_0 \in \mathbb{R}^3$ *and*

$$\|(1-t)x_0 + ty_0\| \le (1-t)\|x_0\| + t\|y_0\| \le 1 - t + t = 1,$$

so $(1-t)x_0 + ty_0 \in S$. *Let*

$$F(t) = (1-t)x_0 + ty_0,$$

and

$$\varphi(t) = f(F(t)) = f((1-t)x_0 + ty_0) \quad \textit{for all } t \in [0,1].$$

Since the functions $F : [0,1] \to S$ *and* $f : S \to \mathbb{R}$ *are continuous,* $\varphi = f \circ F : [0,1] \to \mathbb{R}$ *is continuous.*

Assuming $\alpha < \beta$, *let* $\gamma \in (\alpha, \beta) = (\varphi(0), \varphi(1))$. *Then by the intermediate value property of the continuous function* φ, *there exists* $t_0 \in (0,1)$ *such that*

$$\gamma = \varphi(t_0) = f(F(t_0)) \in f(S),$$

since $F(t_0) \in S$. *Therefore* $f(S) = [\alpha, \beta]$.

Question 0.11 (a)**.** Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{1 - \cos(x^2 + y^2)}{(x^2 + y^2)^2}$$

exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

solution 0.12. *Let* (x_n, y_n) *be any sequence in* $\mathbb{R}^2 \setminus \{(0,0)\}$ *such that* $(x_n, y_n) \to (0,0)$. *Then* $x_n^2 + y_n^2 \ne 0$ *for all* $n \in \mathbb{N}$ *and* $x_n^2 + y_n^2 \to 0$ *in* $\mathbb{R}$. *Since*

$$\lim_{t\to 0} \frac{1 - \cos t}{t^2} = \lim_{t \to 0} \frac{\sin t}{2t} = \frac{1}{2},$$

we have

$$\lim_{n\to\infty} \frac{1 - \cos(x_n^2 + y_n^2)}{(x_n^2 + y_n^2)^2} = \frac{1}{2}.$$

It follows that

$$\lim_{(x,y)\to(0,0)} \frac{1 - \cos(x^2 + y^2)}{(x^2 + y^2)^2}$$

exists and its value is $\frac{1}{2}$.

(b). *Examine whether*

$$\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2}\sin\left(\frac{1}{x^2+y^2}\right)$$

exists (in $\mathbb{R}$*) and find its value if it exists (in* $\mathbb{R}$*).*

solution 0.13. *Let* $f(x,y) = \dfrac{y}{x^2+y^2}\sin\left(\dfrac{1}{x^2+y^2}\right)$ *for all* $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

Since

$$\left(0, \frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}\right) \to (0,0) \quad \textit{and} \quad f\left(0, \frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}\right) = \sqrt{2n\pi + \frac{\pi}{2}} \to \infty,$$

the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *does not exist (in* $\mathbb{R}$*).*

Question 0.14. Let S be a nonempty open set in $\mathbb{R}$ and let $F : S \to \mathbb{R}^m$ be a differentiable function such that $\|F(t)\|$ is constant for all $t \in S$. Show that

$$F(t) \cdot F'(t) = 0 \quad \text{for all } t \in S.$$

solution 0.15. *Let* $c \in \mathbb{R}$ *such that*

$$\|F(t)\| = c \quad \textit{for all } t \in S.$$

Then

$$F(t) \cdot F(t) = \|F(t)\|^2 = c^2 \quad \textit{for all } t \in S.$$

Hence

$$\frac{d}{dt}(F(t) \cdot F(t)) = 0 \quad \textit{for all } t \in S.$$

This gives

$$F'(t) \cdot F(t) + F(t) \cdot F'(t) = 0 \quad \textit{for all } t \in S$$

So

$$2F(t) \cdot F'(t) = 0 \quad \textit{for all } t \in S.$$

Therefore

$$F(t) \cdot F'(t) = 0 \quad \textit{for all } t \in S.$$