MA15010H: Multi-variable Calculus

(Assignment 2: Sequential criteria for continuity and vector differentiability)

July - November, 2025

1. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is (a) an open set (b) a closed set in $\mathbb{R}^3$.

2. Show that $\{x \in \mathbb{R}^m : 1 < \|x\| \leq 2\}$ is neither an open set nor a closed set in $\mathbb{R}^m$.

3. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and if S is a bounded subset of $\mathbb{R}^2$, then $f(S)$ must be a bounded subset of $\mathbb{R}$.

4. Let S be a nonempty subset of $\mathbb{R}^m$ such that every continuous function $f : S \to \mathbb{R}$ is bounded. Show that S is a closed and bounded set in $\mathbb{R}^m$.

5. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ and let $f : S \to \mathbb{R}$ be continuous. Show that there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ such that $f(S) = [\alpha, \beta]$.

6. Examine whether the following limits exist (in $\mathbb{R}$) and find their values if they exist (in $\mathbb{R}$).

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1 - \cos(x^2 + y^2)}{(x^2 + y^2)^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2 + y^2} \sin \frac{1}{x^2 + y^2}$

7. Let S be a nonempty open set in $\mathbb{R}$ and let $F : S \to \mathbb{R}^m$ be a differentiable function such that $\|F(t)\|$ is constant for all $t \in S$. Show that $F(t) \cdot F'(t) = 0$ for all $t \in S$.