Assignment 2

1. Let f be a linear functional on a linear space X. Show that the dimension of quotient space $X/\ker f$ is one.

2. Let f and g be linear functional on linear space X such that $\ker f = \ker g$. Then $f = cg$, for some scalar c.

3. Let f be a linear functional on a normed linear space X. Then f is bounded if and only if $\ker f$ is closed.

4. Let M be a proper subspace of a normed linear subspace X. Suppose $x_o \notin M$ and $\inf_{m\in M} \|x_o - m\| = \delta > 0$. Then there exist $f \in X^*$ such that $\|f\| = 1,\ \ f(x_o) = \delta$ and $f(x) = 0,\ \forall x \in M$. Does such f exist uniquely?

5. Let X^* be the dual space of a normed linear space X. For $x \in X$, show that $\|x\| = \sup\{|f(x)| :\ f \in X^* \text{ and } \|f\| = 1\}$.

6. Suppose X can be made Banach space with respect to norms $\|.\|_1$ and $\|.\|_2$. If there exists $m > 0$ such that $\|x\|_1 \leq m\|x\|_2$ for all $x \in X$. Then both norms are equivalent.

7. Let M be a closed proper subspace of a normed linear space X. Then M is nowhere dense in X.

8. Let $X = (C[0,1], \|.\|_\infty)$. For $f \in C[0,1]$ define
$$T(f)(t) = \int_{s=0}^{t} f(s)ds.$$
Then show that for each $n \in \mathbb{N}$, the power of operator T satisfies $\|T^n\| = \frac{1}{n!}$.

9. Let $C_c(\mathbb{R})$ be the class of all compactly supported continuous functions on $\mathbb{R}$. Does the linear functional given by
$$T(f) = \int_{-\infty}^{\infty} f(t)dt$$
is continuous? What happen when T is defined on $C_c(\mathbb{C})$ with respect to complex measure on $\mathbb{C}$?

10. Let X and Y be two Banach spaces and $T : X \to Y$ is a continuous linear bijection. Then show that $\inf\{\|x\| : \|Tx\| = 1\} = \|T^{-1}\|$.

11. Prove that an infinite dimensional Banach space X can not be expressed as the countable union of compact subsets of X.

12. Let X be a Banach space and Y be a normed linear space. Suppose $T_n \in B(X,Y)$ such that $\lim_{n\to\infty} T_n(x)$ exists for each $x \in X$. Write $T(x) = \lim_{n\to\infty} T_n(x)$. Show that T is bounded.

13. Let X and Y be two normed linear spaces and $T_n, T \in B(X,Y)$ such that $T_n \to T$. Suppose $x_n \to x$. Then show that $T_n x_n \to Tx$ in Y.

14. Show that $\{(x_n) \in l_2 : \ |x_n| \leq \frac{1}{n}\}$ is a compact convex set of l^2 with empty interior.

15. Let X and Y be two normed linear spaces and $T \in B(X,Y)$ sends each open subset in X to an open subset in Y. Prove that T is onto.

16. Let X and Y be two Banach spaces and $T \in B(X,Y)$. Show that followings are equivalent:
(a). T is injective and has closed range.
(b). There is $k \geq 0$ such that $\|x\| \leq k\|T(x)\|$ for all $x \in X$.

17. Let X and Y be two normed linear spaces and $T : X \to Y$ such that $\dim T(X) < \infty$ and $\ker T$ is closed. Then show that T is bounded.

18. Let X and Y be two Banach spaces and $T_n, T \in B(X,Y)$ such that $T_n \to T$ weakly. Then $\sup_n \|T_n\| < \infty$.

19. Let X and Y be two Banach spaces and $T : X \to Y$ such that $f \circ T \in X^*$ for each $f \in Y^*$. Then T is continuous.

20. Let X and Y be two Banach spaces and $S : X \to Y$ and $T : Y^* \to X^*$ are linear maps such that $f \circ S = T(f)$, for all $f \in Y^*$. Then show that S continuous. (Hint: use close graph theorem).

21. Let $X = c_o$. Show that $X^* = l^1$ and $X^{**} = l^\infty$. For $x = (x_n) \in X$, prove that $x \mapsto \sum_1^\infty x_n$ is weakly continuous but not weak* continuous.

22. Let X be a reflexive Banach space and $f \in X^*$. Show that there exists $x_o \in \overline{B(0,1)}$ such that $f(x_o) = \|f\|$.

23. Let K be a closed bounded convex subset of a reflexive Banach space X. Prove that K is weakly compact.

24. Suppose M is a subspace of a Banach space X. Then $M^\perp$ is weak* closed subspace of X^*.