# Assignment 1

1. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $1 \leq p < q \leq \infty$, prove that
$$\|x\|_q \leq \|x\|_p \leq n^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|x\|_q.$$

2. Let $\mathbb{Q} = \{r_1, r_2, \ldots, \}$ be an enumeration set of rational numbers. Define a sequence of functions $f_n : [0,1] \to \mathbb{R}$ by
$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \cap [0,1], \\ 0, & \text{otherwise.} \end{cases}$$
Show that $(f_n)$ is a Cauchy sequence in $(\mathcal{R}[0,1], \| \, . \, \|_1)$ but it does not converge to a function in $\mathcal{R}[0,1]$. Does $(f_n)$ converge in $L^1[0,1]$?

3. Let $C^1[0,1]$ denote the space of all continuously differentiable functions on $[0,1]$. For $f \in C^1[0,1]$, define $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Show that space $(C^1[0,1], \| \, . \, \|)$ is a Banach space.

4. Does space $(C^1[0,1], \| \, . \, \|)$ , where $\|f\| = \|f'\|_1 + \|f\|_1$ a Banach space?

5. Let $f \in C^1[0,1]$. Write $\|f\| = \|f\|_2 + \|f'\|_\infty$. Whether $(C^1[0,1], \| \, . \, \|)$ is a Banach space?

6. Let $f \in C^1[0,1]$. Does $\|f\| = \min\left(\|f'\|_2, \ \|f\|_\infty\right)$ defines a norm on $C^1[0,1]$?

7. Let $X = \{f \in C^1[0,1] : \ f(0) = 0\}$. Then $\|f\| = \left(\int_0^1 |f'|^2\right)^{\frac{1}{2}}$ defines a norm on $C^1[0,1]$. Whether $(X, \ \| \, . \, \|)$ is a Banach space ?

8. Let $(V, \ \| \, . \, \|)$ be a Banach space and $X$ is the space of all the continuous functions from $[0,1]$ to $V$ with $\|f\|_\infty = \sup_{t \in [0,1]} \|f(t)\|$. Prove that $(X, \ \| \, . \, \|_\infty)$ is a Banach space.

9. Show that $L^p[0,1]$ is proper dense subspace of $L^1[0,1]$, whenever $1 < p < \infty$.

10. Let $C_o$ be the class of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $|f(x)| < \epsilon$, for all $x \in \mathbb{R} \setminus K$. Show that $(C_o, \ \| \, . \, \|_\infty)$ is a Banach space.

11. Let $C_c(\mathbb{R}^n)$ denotes the class of all compactly supported continuous functions on $\mathbb{R}^n$.

    **(i)** Prove that $C_c(\mathbb{R})$ is a proper dense subspace of $L^p(\mathbb{R})$, whenever $1 \leq p < \infty$.

    **(ii)** Whether $C_c(\mathbb{R})$ is a dense subspace of $L^\infty(\mathbb{R})$?

**(iii)** Prove that $C_c(\mathbb{R})$ is a dense subspace of $(C_o, \ \| \ . \ \|_\infty)$ .

**(iii)** Prove that $L^1 \cap L^p(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, whenever $1 < p < \infty$.

12. Let $1 \leq p < q < \infty$. Prove that $L^q[0,1]$ is a dense proper subspace of $L^p[0,1]$.

13. Let $S(\mathbb{R})$ be the space of simple functions on $\mathbb{R}$. Prove that $S(\mathbb{R})$ is dense in $Ł^p(\mathbb{R})$, for $1 \leq p < \infty$. Whether $S(\mathbb{R})$ is dense in $Ł^\infty(\mathbb{R})$?

14. Let $(x_n)$ be a sequence in a normed linear space $X$ which converges to a non-zero vector $x \in X$. Show that
$$\frac{x_1 + \cdots + x_n}{n^\alpha} \to x$$
if and only if $\alpha = 1$. If the sequence $x_n \to 0$, prove that
$$\frac{x_1 + \cdots + x_n}{n^\alpha} \to 0, \ \text{for all } \alpha \geq 1.$$

15. Let $M$ be a subspace of a normed linear space $X$. Then show that $M$ is closed if and only if $\{y \in M : \ \|y\| \leq 1\}$ is closed in $X$.

16. Let $D = \{z \in \mathbb{C} : \ |z| < 1\}$. Let $X$ be the class of all functions $f$ which are analytic on $D$ and continuous on $\bar{D}$. Define $\|f\|_\infty = \sup\{|f(e^{it})| : \ 0 \leq t \leq 2\pi\}$. Prove that $(X, \ \| \ . \ \|_\infty)$ is a Banach space.

17. Show that unit ball $\{x \in l^1(\mathbb{N}) : \ \|x\|_1 \leq 1\}$ is not compact in $l^1(\mathbb{N})$.

18. Show that unit ball $\{x \in l^\infty(\mathbb{N}) : \ \|x\|_\infty \leq 1\}$ is not compact $l^\infty(\mathbb{N})$.

19. Whether the annulus $\{x \in l^2(\mathbb{N}) : \ 1 \leq \|x\|_2 \leq 2\}$ is a compact set in $l^2(\mathbb{N})$?

20. A normed linear space $X$ is finite dimensional if and only if any ball $B_r(x)$ in $X$ is compact.

21. Let $M$ be a closed subspace of a normed linear space $X$. Prove that projection $\pi : \ X \to X/M$ defined by $\pi(x) = \tilde{x}$ is a continuous map.

22. Let $X$ be a normed linear space. Prove that norm of any $x \in X$, can be expressed as $\|x\| = \inf\{|\alpha| : \ \alpha \in \mathbb{C} \setminus \{0\} \text{ with } \|x\| \leq |\alpha|\}$ .

23. Prove that $L^p(\mathbb{R})$ is separable for $1 \leq p < \infty$ but $L^\infty(\mathbb{R})$ is not separable.

24. Let $M$ be a closed subspace of a normed linear space $X$. Then show that $X$ is separable if and only if $M$ and $X/M$ both are separable.

25. Let $X$ be a separable Banach space. Prove that there exists a closed subspace $M$ of $l^1(\mathbb{N})$ such that $X$ is isomorphic to $l^1(\mathbb{N})/M$.