MA746: Fourier Analysis

(Assignment 1: Fourier Series)

July - November, 2025

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If D_n is the sequence of Dirichlet kernel on S^1. Then it necessarily implies that $D_n * D_n = D_n$.
   (b) Does there exist $f \in L^1(S^1)$ such that $\sum\limits_{n=-\infty}^{\infty} |n\hat{f}(n)|^2 = \infty$?

2. If f is continuously differentiable on S^1, then show that $\hat{f'}(n) = in\hat{f}(n)$ for all $n \in \mathbb{Z}$. Deduce that that there exists $C > 0$ such that $|\hat{f}(n)| \leq \frac{C}{|n|}$. Does the above conclusion hold if f is absolutely continuous?

3. If f is a function of bounded variation on $[-\pi, \pi]$, then show that $|\hat{f}(n)| \leq \frac{\mathrm{Var}(f)}{2\pi|n|}$ for all $n \in \mathbb{Z}$.

4. For $f \in L^1(S^1)$, show that $\hat{f}(n) = \frac{1}{4\pi} \int\limits_{-\pi}^{\pi} \left[f(x) - f(x + \frac{\pi}{n})\right] e^{-inx} dx$. Further, use this identity to prove the Riemann-Lebesgue lemma.

5. Let $f \in L^1(S^1)$ be such that $|f(x+h) - f(x)| \leq M|h|^\alpha$, for some $0 < \alpha < 1$ and $M > 0$ and for all $x, h \in S^1$. Show that $\hat{f}(n) = O\left(\frac{1}{|n|^\alpha}\right)$.

6. Show that the Fejer's kernel F_n can be expressed as $F_n(t) = \sum\limits_{j=-n}^{n} \left(1 - \frac{|j|}{n}\right) e^{ijt}$.

7. Let $f \in L^1(S^1)$ and $m \in \mathbb{N}$ and define $f_m(t) = f(mt)$. Prove that $\hat{f}_m(n) = \hat{f}(\frac{n}{m})$, if $(m, n) \neq 1$ and $\hat{f}_m(n) = 0$ otherwise.

8. Let f be a function on S^1. For $x, y \in S^1$, define $\tau_x f(y) = f(x - y)$. Show that $x \to \tau_x f$ is continuous in $L^p(S^1)$ for $1 \leq p < \infty$. That is, $\|\tau_x f - f\|_p \to 0$ when $|x| \to 0$. Does the above conclusion hold if $p = \infty$?

9. If $f \in L^1(S^1)$ and $g \in L^\infty(S^1)$, then show that $\lim\limits_{n\to\infty} \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(t)g(nt)dt = \hat{f}(0)\hat{g}(0)$.

10. For given $f \in L^1(S^1)$, define an operator T_f on $L^1(S^1)$ by $T_f(g) = f * g$. Show that T_f is a bounded operator on $L^1(S^1)$ and $\|T_f\| = \|f\|_1$.

11. Let P be a trigonometric polynomial of degree n on S^1. Show that $\|P'\|_\infty \leq 2n\|P\|_\infty$.

12. Let $1 \leq p \leq \infty$ and $p^{-1} + q^{-1} = 1$. For $f \in L^p(S^1)$ and $g \in L^q(S^1)$, show that $f * g$ is a continuous function on S^1.

13. Suppose $f \in L^\infty(S^1)$ satisfies $|\hat{f}(n)| \leq \frac{k}{|n|}$ for some $k > 0$ and for all $n \in \mathbb{Z} \setminus \{0\}$. Prove that $|S_n(f)(t)| \leq \|f\|_\infty + 2k$, where $S_n(f) = D_n * f$.

14. If f is a bounded monotone function on S^1, then show that $\hat{f}(n) = O\left(\frac{1}{|n|}\right)$.

15. If f is Riemann integrable on $[-\pi, \pi]$, then show that $\sum\limits_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty$ and hence $\hat{f}(n) = o(1)$.

16. Prove that, if the series of complex numbers $\sum_{n=0}^{\infty} a_n$ converges to s, then $\sum_{n=0}^{\infty} a_n$ is Cesaro as well as Abel summable to s.

17. Prove that, if the series of complex numbers $\sum_{n=0}^{\infty} a_n$ is Cesaro summable to σ, then $\sum_{n=0}^{\infty} a_n$ is Abel summable to σ. However, converse need not be true.

18. If the series of complex numbers $\sum_{n=0}^{\infty} a_n$ is Cesaro summable to l, then show that $\lim_{n\to\infty} \frac{a_n}{n} = 0$, where $s_n = a_1 + \cdots + a_n$.

19. Let $u(r,\theta) = \frac{\partial P_r}{\partial \theta}(\theta)$, where $P_r(\theta)$ is the Poisson kernel defined on the open unit disc $\mathbb{D} = \{re^{i\theta} : 0 \le r < 1, \theta \in [-\pi,\pi)\}$. Show that $\Delta u = 0$ on $\mathbb{D}$ and $\lim_{r\to 1} u(r,\theta) = 0$ for all $\theta \in [-\pi,\pi)$.

20. Let f be Riemann integrable on $[-\pi,\pi]$, and $A_r(f)(\theta) = f * P_r(\theta)$ for $0 \le r < 1$, denotes the Abel mean of f. If f has jump discontinuity at θ, then show that $\lim_{r\to 1} A_r(f)(\theta) = \frac{f(\theta^+)+f(\theta^-)}{2}$. Justify that $\lim_{r\to 1} A_r(f)(\theta) \neq \frac{f(\theta)}{2}$, when f is continuous at θ.

21. Let f be Riemann integrable on $[-\pi,\pi]$, and $\sigma_n(f)(\theta) = f * F_n(\theta)$, where F_n is Fejer's kernel. If f has jump discontinuity at θ, then show that $\lim_{n\to\infty} \sigma_n(f)(\theta) = \frac{f(\theta^+)+f(\theta^-)}{2}$.

22. Let f be Riemann integrable on $[-\pi,\pi]$ such that $\hat{f}(n) = O\left(\frac{1}{|n|}\right)$ for all $n \in \mathbb{Z}$.
(a) Show that $S_N(f)(\theta) = D_N * f(\theta) \to f(\theta)$ if f is continuous at θ.
(b) If f has jump discontinuity at θ, then show that $S_N(f)(\theta) \to \frac{f(\theta^+)+f(\theta^-)}{2}$.
(c) If f is continuous on $[-\pi,\pi]$, then $S_N(f) \to f$ uniformly.

23. Let f be Lebesgue measurable function on S^1 such that $\int_0^{2\pi} \frac{|f(t)|}{t} dt < \infty$. Show that $\lim_{n\to\infty} S_n(f;0) = 0$.

24. For $f \in L^2(S^1)$, show that $\frac{1}{n}\sum_{k=0}^{n-1} f(x+\frac{k}{n}) \to \hat{f}(0)$ as $n \uparrow \infty$ in the metric of $L^2(S^1)$.

25. Does there exist a function $f \in L^1(S^1)$ such that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \infty$?

26. If $f \in L^1(S^1)$ vanishes near $x=0$, then show that $S_N(f) \to 0$ uniformly near $x = 0$.

27. Let f be a function on $[-\pi,\pi]$ such that $|f(\theta) - f(\varphi)| \le M|\theta - \varphi|$, for some $M > 0$ and for all $\theta, \varphi \in [-\pi,\pi]$.
(a) For $u(r,\theta) = P_r * f(\theta)$, show that $\frac{\partial u}{\partial \theta}$ exists for all $0 \le r < 1$ and $\left|\frac{\partial u}{\partial \theta}\right| \le M$ for some $M > 0$.
(b) Show that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \le |\hat{f}(0)| + 2M\sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}}$.

28. If f is continuously differentiable on S^1, then show that $\sum_{n=-\infty}^{\infty} (1+|n|^2)|\hat{f}(n)|^2 < \infty$.

29. If $\{G_n\}_{n=1}^{\infty}$ is a family of good kernels on S^1, then show that $\lim_{n\to\infty} \hat{G}_n(k) = 1$.

30. Let f and g be Riemann integrable on $[-\pi,\pi]$. Define $\tilde{g}(x) = \overline{g(-x)}$.
(a) Show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(t)|^2 dt = g * \tilde{g}(0)$.

(b) $\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |f * g(x)|^2 dx = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |f * \tilde{g}(x)|^2 dx.$

31. Let $f \in L^1(S^1)$ be such that $\hat{f}(|n|) = -\hat{f}(-|n|) \geq 0$ for all $n \in \mathbb{Z}$. Show that $\sum\limits_{n>0} \frac{\hat{f}(n)}{n} < \infty$.

32. If $\{K_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\infty}$ are families of good kernels on S^1, then show that $\{K_n * J_n\}_{n=1}^{\infty}$ is a family of good kernels.

33. Suppose f is an absolutely continuous function on S^1 such that $f' \in L^2(S^1)$. Prove that $\sum\limits_{n=-\infty}^{\infty} |\hat{f}(n)| \leq \|f\|_1 + 2\sqrt{\sum\limits_{n=1}^{\infty} \frac{1}{n^2}}\|f'\|_2$.

34. Show that there exists $f \in L^1(S^1)$ such that partial sum sequence $S_n(f)$ of the Fourier series of f does not converge to f in L^1-norm.

35. Let $f \in L^1(S^1)$ and $S_n(f)$ denotes the nth partial sum of the Fourier series of f. Show that $\left\|\frac{S_n(f)}{n}\right\|_1 \to 0$ as $n \to \infty$.

36. Let f be a Riemann integrable function on $[-\pi, \pi]$. If f is differentiable at $t_o \in [\pi, \pi]$ then show that $S_n(f; t_o) \to f(t_o)$ as $n \to \infty$.

37. Suppose $f \in C^1(S^1)$ is satisfying $[f * (1 + f)](t) = f'(t)$ for all $t \in S^1$. Show that f is constant.