Advanced Course on Hardy spaces

(MA650: Assignment 1)

January- April, 2022

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) Every subspace of $L^2(\mathbb{T}, m)$ of dimension more than one is simply invariant.
   (b) Let $H^2 = \overline{\text{span}}\{z^n : n \geq 0\}$. Whether $H^2 \perp zH^2$?
   (c) If $0 \neq f \in H^2$, then $E_f = \overline{\text{span}}\{z^n f : n \geq 0\}$ is a reducing subspace of H^2.
   (d) Let μ be a finite measure on $\mathbb{T}$. Then E_f is always a reducing subspace of $L^2(\mu)$?
   (e) If $\Theta \in H^2$ is an inner function, does it imply that $\overline{\text{span}}\{z^n\Theta : n \geq 0\} = \Theta H^2$?
2. Let μ be a finite Borel measure on $\mathbb{T}$. Prove or disprove that $L^2(\mu) = L^2(\mu).L^2(\mu)$.
3. Let μ be a finite Borel measure on $\mathbb{C}$. Prove or disprove that for every $f \in L^2(\mathbb{C}, \mu)$ there exist $g, h \in L^2(\mathbb{C}, \mu)$ such that $f = gh$.
4. Let $w \in L^1_+(\mathbb{T}, m) = \{g \in L^1(\mathbb{T}, m) : g \geq 0\}$. If there exits $f \in H^2$ sauch that $|f|^2 = w$ a.e. on $\mathbb{T}$. Then there exists a unique outer function f_o such that $|f_o|^2 = w$ a.e. on $\mathbb{T}$.
5. Let μ be a finite Borel measure on $\mathbb{T}$. Write $H^2_0(\mu) = zH^2(\mu)$. Show that $H^2_0(\mu) = H^2_0(\mu_a) \oplus L^2(\mu_s)$, where $\mu = \mu_a + \mu_s$.
6. Let μ be a finite Borel measure on $\mathbb{T}$. Then following are equivalent:
   (i) There exists a non-reducing subspace $E \subset L^2(\mu)$ with $zE \subset E$.
   (ii) There exists a complex measure $\nu \neq 0$ which is absolutely continuous with respect to μ and orthogonal to $\mathbb{P}_+$, i.e. $\int\limits_{\mathbb{T}} z^n d\nu \;\forall\; n \geq 1$.
7. Let μ be a finite measure on $\mathbb{T}$. Then $zE \subseteq E \subset L^2(\mu)$ implies $zE = E$ if and only if m is not absolutely continuous with respect to μ.
8. Let μ be a compactly supported finite measure on the complex plane $\mathbb{C}$. Show that every reducing subspace E of $L^2(\mu)$ is of the form $E = \chi_\sigma L^2(\mu)$ for a Borel set σ of $\mathbb{C}$.
9. Let $L^\infty(\mathbb{T}, m)$ be the space essentially bounded measurable functions on $\mathbb{T}$.
   (i) If $f \in H^2 \cap L^\infty$, then $fH^2 \subset H^2$.
   (ii) If $f \in H^2 \cap L^\infty$ and $\|f\|_\infty < 1$, then $1 + f$ is outer
   (iii) If $f \in H^2 \cap L^\infty$, then $e^f \in H^2$ is an outer function.
10. For $\lambda \in \mathbb{C}$, $z - \lambda$ is an outer function if and only if $|\lambda| \geq 1$. Hence polynomial p is an outer function if and only if p has no zero in the open unit disc $\mathbb{D} = \{z \in: |z| < 1\}$.
11. Let μ be a finite measure on $\mathbb{T}$. If $H^2(\mu)$ is a proper subspace of $L^2(\mu)$, then show that $\text{dist}(1, H^2_o(\mu)) > 0$.
12. If $f \in H^2$ be an outer function, then show that $\overline{\text{span}}\{z^n f : n \geq 1\} = zH^2$.