

MA642: Real Analysis -1

(Assignment 1: Metric and Normed Linear Spaces)

July - November, 2025

1. State TRUE or FALSE giving proper justification for each of the following statements.
 - (a) There does not exist a monotone function $f : \mathbb{R} \rightarrow \mathbb{Q}$ which is onto.
 - (b) There exists a monotone function $f : (0, \infty) \rightarrow \mathbb{R}$ such that each $c \in (0, \infty)$ satisfies $|f(c+) - f(c-)| = \frac{1}{c}$.
 - (c) There exists a sequence of differential functions f_n on $(0, \infty)$ such that f'_n is uniformly convergent on $(0, \infty)$ but f_n is nowhere point-wise convergent.
 - (d) There exists a metric space having exactly 36 open sets.
 - (e) It is impossible to define a metric d on \mathbb{R} such that only finitely many subsets of \mathbb{R} are open in (\mathbb{R}, d) .
 - (f) If A and B are open (closed) subsets of a normed vector space X , then $A + B = \{a + b : a \in A, b \in B\}$ is open (closed) in X .
 - (g) If A and B are closed subsets of $[0, \infty)$ (with the usual metric), then $A + B$ is closed in $[0, \infty)$.
 - (h) It is possible to define a metric d on \mathbb{R} such that the sequence $(1, 0, 1, 0, \dots)$ converges in (\mathbb{R}, d) .
 - (i) It is possible to define a metric d on \mathbb{R}^2 such that $((\frac{1}{n}, \frac{n}{n+1}))$ is not a Cauchy sequence in (\mathbb{R}^2, d) .
 - (j) It is possible to define a metric d on \mathbb{R}^2 such that in (\mathbb{R}^2, d) , the sequence $((\frac{1}{n}, 0))$ converges but the sequence $((\frac{1}{n}, \frac{1}{n}))$ does not converge.
 - (k) Let $A \subset (1, \infty)$ be a closed set. Then $A^2 := \{a^2 : a \in A\}$ is a closed set.
 - (l) Let $A_n = \{(x, y) \in \mathbb{R}^2 : 0 < \frac{1}{x} < y < \frac{1}{n}\}$. Whether the set $\bigcap_{n=1}^{\infty} A_n$ is open/closed?
 - (m) There exist a set $A \subset (\mathbb{R}, u)$ such that $\delta(A^o \cup \{0\}) = 0$ but $\delta((\bar{A})^o) = 1$, where δ stands for diameter.
 - (n) If (x_n) is a sequence in a complete normed vector space X such that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$, then (x_n) must converge in X .
 - (o) If (f_n) is a sequence in $C[0, 1]$ such that $|f_{n+1}(x) - f_n(x)| \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$, then there must exist $f \in C[0, 1]$ such that $\int_0^1 |f_n(x) - f(x)| dx \rightarrow 0$ as $n \rightarrow \infty$.
 - (p) If (x_n) is a Cauchy sequence in a normed vector space, then $\lim_{n \rightarrow \infty} \|x_n\|$ must exist.
 - (q) $\{f \in C[0, 1] : \|f\|_1 \leq 1\}$ is a bounded subset of the normed vector space $(C[0, 1], \|\cdot\|_{\infty})$.
 - (r) For $x, y \in l^{\infty}$, $d(x, y) = \min\{1, \limsup |x_n - y_n|\}$ define a metric on l^{∞} .
 - (s) The sequence $f_n(t) = e^{-n^2 \sin \pi t}$ converge uniformly to 0 on $(0, 1)$.
 - (t) If sequence (x_n) in \mathbb{R} satisfies $0 \leq \inf x_n \sup x_n < \infty$, then x_n has necessarily convergent subsequence. item If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, bounded and monotone function, then $\lim_{x \rightarrow \pm\infty} f(x)$ are necessarily finite?
2. What is the cardinality of the set $\{f : \mathbb{R} \rightarrow \mathbb{R}, f \text{ is nowhere continuous}\}$?
3. For a monotone increasing function $f : [a, b] \rightarrow \mathbb{R}$, define $g(x) = \sup\{f(y) : y < x\}$. If f has limit at c , then show that $f(c) = g(c)$.
4. Examine whether (X, d) is a metric space, where
 - (a) $X = \mathbb{R}$ and $d(x, y) = \frac{|x-y|}{1+|xy|}$ for all $x, y \in \mathbb{R}$.
 - (b) $X = \mathbb{R}$ and $d(x, y) = |x - y|^p$ for all $x, y \in \mathbb{R}$ ($0 < p < 1$).
 - (c) $X = \mathbb{R}$ and $d(x, y) = \min\{\sqrt{|x - y|}, |x - y|^2\}$ for all $x, y \in \mathbb{R}$.

- (d) $X = \mathbb{R}$ and for all $x, y \in \mathbb{R}$, $d(x, y) = \begin{cases} 1 + |x - y| & \text{if exactly one of } x \text{ and } y \text{ is positive,} \\ |x - y| & \text{otherwise.} \end{cases}$
- (e) $X = \mathbb{R}^2$ and $d(x, y) = (|x_1 - y_1| + |x_2 - y_2|)^{\frac{1}{2}}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.
- (f) $X = \mathbb{R}^n$ and $d(x, y) = [(x_1 - y_1)^2 + \frac{1}{2}(x_2 - y_2)^2 + \cdots + \frac{1}{n}(x_n - y_n)^2]^{\frac{1}{2}}$ for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$.
- (g) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} \min\{|z| + |w|, |z - 1| + |w - 1|\} & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$
- (h) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} |z - w| & \text{if } \frac{z}{|z|} = \frac{w}{|w|}, \\ |z| + |w| & \text{otherwise.} \end{cases}$
- (i) $X = \mathbb{C}$ and $d(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}$ for all $z, w \in \mathbb{C}$.
- (j) $X =$ The class of all finite subsets of a nonempty set and $d(A, B) =$ The number of elements of the set $A \triangle B$ (the symmetric difference of A and B).

5. Let $1 \leq p \leq \infty$ and $d_i; i = 1, 2$ be two metric on a non-empty set X . Show that $d_p = (d_1^p + d_2^p)^{1/p}$ is a metric on X for $1 \leq p < \infty$. Whether $d_\infty = \max\{d_1, d_2\}$ is a metric on X ?
6. Examine whether $\|\cdot\|$ is a norm on \mathbb{R}^2 , where for each $(x, y) \in \mathbb{R}^2$,
- (a) $\|(x, y)\| = (|x|^p + |y|^p)^{\frac{1}{p}}$, where $0 < p < 1$.
- (b) $\|(x, y)\| = \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}$.
- (c) $\|(x, y)\| = \begin{cases} \sqrt{x^2 + y^2} & \text{if } xy \geq 0, \\ \max\{|x|, |y|\} & \text{if } xy < 0. \end{cases}$
7. Let $\|f\| = \min\{\|f\|_\infty, 2\|f\|_1\}$ for all $f \in C[0, 1]$. Prove that $\|\cdot\|$ is not a norm on $C[0, 1]$.
8. Let X be a normed linear space. Prove that norm of any $x \in X$, can be expressed as $\|x\| = \inf\{|\alpha| : \alpha \in \mathbb{C} \setminus \{0\} \text{ with } \|x\| \leq |\alpha|\}$.
9. Let $(X, \|\cdot\|)$ be a normed linear space. Show that $\|x\| = \sup\{|\alpha| : |\alpha| < \|x\|\}$.
10. Let $(X, \|\cdot\|)$ be a normed linear space and let p be a seminorm on X . Show that $p : (X, \|\cdot\|) \rightarrow \mathbb{R}$ is continuous iff there exists $\alpha > 0$ such that $p(x) \leq \alpha\|x\|$ for all $x \in X$.
11. If $1 \leq p < q \leq \infty$, then show that $\|x\|_q \leq \|x\|_p$ for all $x \in \ell^p$.
12. If $\mathbf{x} \in \mathbb{R}^n$, then show that $\lim_{p \rightarrow \infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$. And if $x \in \ell^p$, then show that $\liminf_{p \rightarrow \infty} \|\mathbf{x}\|_p \geq \|\mathbf{x}\|_\infty$.
13. Let d be a metric on a real vector space X satisfying the following two conditions:
- (i) $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$,
- (ii) $d(\alpha x, \alpha y) = |\alpha|d(x, y)$ for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$.
- Show that there exists a norm $\|\cdot\|$ on X such that $d(x, y) = \|x - y\|$ for all $x, y \in X$.
14. Let f be a non-negative function on a linear space X such that $f(\alpha x) = |\alpha|f(x)$ for all $\alpha \in \mathbb{C}$. Show that f is norm on X if and only if f is a convex map which can vanish at most at one point.
15. Let $f : (X, d) \rightarrow [0, 1]$ be continuous map. Show that $f^{-1}(0)$ is a closed G_δ set.
16. Let (x_n) be a sequence in a normed linear space X which converges to a non-zero vector $x \in X$. Show that $\frac{x_1 + \cdots + x_n}{n^\alpha} \rightarrow x$ if and only if $\alpha = 1$. What are admissible values of α if $x_n \rightarrow 0$?
17. For $x = (x_n) \in \ell^2$, write $\|x\| = (\sum_{n=1}^{\infty} a_n |x_n|^2)^{1/2}$. Find all possible sequences (a_n) such that $\|\cdot\|$ is a norm on ℓ^2 .
18. Let \mathbb{R}^∞ be the real vector space of all sequences in \mathbb{R} , where addition and scalar multiplication are defined componentwise. Let $d((x_n), (y_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}$ for all $(x_n), (y_n) \in \mathbb{R}^\infty$. Show that d is a metric on \mathbb{R}^∞ but that no norm on \mathbb{R}^∞ induces d .

19. Let $(X, \|\cdot\|)$ be a nonzero normed vector space. Consider the metrics d_1, d_2 and d_3 on X :

$$d_1(x, y) := \min\{1, \|x - y\|\},$$

$$d_2(x, y) := \frac{\|x - y\|}{1 + \|x - y\|},$$

$$d_3(x, y) := \begin{cases} 1 + \|x - y\| & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. Prove that none of d_1, d_2 and d_3 is induced by any norm on X .

20. Let X be a normed vector space containing more than one point, let $x, y \in X$ and let $\varepsilon, \delta > 0$. If $B_\varepsilon[x] = B_\delta[y]$, show that $x = y$ and $\varepsilon = \delta$. Does the result remain true if X is assumed to be a metric space? Justify.
21. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is a closed/an open subset of \mathbb{R}^3 with respect to the usual metric on \mathbb{R}^3 .
22. Let F_n be a sequence of closed sets in \mathbb{R} such that $F_n \subset (n, n+1]$ and $F_n \cap F_m = \emptyset$, whenever $m \neq n$. Show that $F = \bigcup_{n=1}^{\infty} F_n$ is a closed set in \mathbb{R} .
23. For all $x, y \in \mathbb{R}$, let $d_1(x, y) = |x - y|$, $d_2(x, y) = \min\{1, |x - y|\}$ and $d_3(x, y) = \frac{|x-y|}{1+|x-y|}$. If G is an open set in any one of the three metric spaces (\mathbb{R}, d_i) ($i = 1, 2, 3$), then show that G is also open in the other two metric spaces.
24. Let X be a normed vector space and let $Y (\neq X)$ be a subspace of X . Show that Y is not open in X .
25. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d) . Show that the sequence $(d(x_n, y_n))$ is convergent.
26. Let d_o be the discrete metric on non-empty set X . Show that (X, d_o) is complete.
27. Let (x_n) be a sequence in a complete metric space (X, d) such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. Show that (x_n) converges in (X, d) .
28. Let (x_n) be a sequence in a metric space X such that each of the subsequences $(x_{2n}), (x_{2n-1})$ and (x_{3n}) converges in X . Show that (x_n) converges in X .
29. Show that the following are incomplete metric spaces.
- (\mathbb{N}, d) , where $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$ for all $m, n \in \mathbb{N}$
 - $((0, \infty), d)$, where $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$ for all $x, y \in (0, \infty)$
 - (\mathbb{R}, d) , where $d(x, y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$ for all $x, y \in \mathbb{R}$
 - (\mathbb{R}, d) , where $d(x, y) = |e^x - e^y|$ for all $x, y \in \mathbb{R}$
30. Examine whether the following metric spaces are complete.
- $([0, 1), d)$, where $d(x, y) = |\frac{x}{1-x} - \frac{y}{1-y}|$ for all $x, y \in [0, 1)$
 - $((-1, 1), d)$, where $d(x, y) = |\tan \frac{\pi x}{2} - \tan \frac{\pi y}{2}|$ for all $x, y \in (-1, 1)$
 - $((0, 2], d)$, where $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$
31. For $X (\neq \emptyset) \subset \mathbb{R}$, let $d(x, y) = \frac{|x-y|}{1+|x-y|}$ for all $x, y \in X$. Examine the completeness of the metric space (X, d) , where X is
- $[0, 1] \cap \mathbb{Q}$.
 - $[-1, 0] \cup [1, \infty)$.
 - $\{n^2 : n \in \mathbb{N}\}$.

32. Let $X = C[0, 1]$ be the space all the continuous functions on interval $[0, 1]$. Prove that norms $\| \cdot \|_\infty$ and $\| \cdot \|_1$ on X are not equivalent.
33. Let $C^1[0, 1]$ denote the space of all continuously differentiable functions on $[0, 1]$. For $f \in C^1[0, 1]$, define $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Show that space $(C^1[0, 1], \| \cdot \|)$ is a Banach space.
34. The space $(C^1[0, 1], \| \cdot \|)$, where $\|f\| = (\|f\|_2^2 + \|f'\|_2^2)^{\frac{1}{2}}$ is complete.
35. Let $f \in C^1[0, 1]$ and $\|f\| = \|f'\|_2 + \|f\|_\infty$. Then verify if $(C^1[0, 1], \| \cdot \|)$ is complete.
36. Let $f \in C^1[0, 1]$. Then verify if $\|f\| = \min(\|f'\|_2, \|f\|_\infty)$ defines a norm on $C^1[0, 1]$.
37. Let $X = \{f \in C^1[0, 1] : f(0) = 0\}$. Then $\|f\| = \|f'\|_2$ is a norm on $C^1[0, 1]$ but not complete.
38. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Let X be the class of all functions f which are analytic on D and continuous on \bar{D} . Define $\|f\| = \sup\{|f(e^{it})| : 0 \leq t \leq 2\pi\}$. Show that $(X, \| \cdot \|)$ is complete.
39. Examine whether the sequence (f_n) is convergent in $(C[0, 1], d_\infty)$, where for all $n \in \mathbb{N}$ and for all $t \in [0, 1]$,
- $f_n(t) = \frac{nt^2}{1+nt}$.
 - $f_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$.
 - $f_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n}, \\ \frac{1}{nt} & \text{if } \frac{1}{n} < t \leq 1. \end{cases}$
 - $f_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n}, \\ \frac{n}{n-1}(1-t) & \text{if } \frac{1}{n} < t \leq 1. \end{cases}$
40. Find the point-wise limit of the sequence $f_n(t) = e^{-nt^2} \sin nt$. Examine for uniform convergence of f_n on \mathbb{R} .
41. Let $f_n, f : \mathbb{R} \rightarrow (0, \infty)$ be such that $f_n \rightarrow f$ uniformly on \mathbb{R} . Examine for $e^{f_n} \rightarrow e^f$ uniformly on \mathbb{R} .
42. Let $f_n(t) = \sqrt{t^2 + n}$. Examine for the uniform convergence of f'_n on \mathbb{R} .
43. Let X be the class of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{C}$ such that for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $|f(x)| < \epsilon$, for all $x \in \mathbb{R} \setminus K$. Show that $(X, \| \cdot \|_\infty)$ is complete.
44. Let $1 \leq p < \infty$. Let X_p be a class of all the Riemann integrable functions on $[0, 1]$. Prove that $\|f\|_p = \left(\int_0^1 |f|^p\right)^{\frac{1}{p}} < \infty$. Prove that $(X_p, \| \cdot \|_p)$ is a normed linear space but not complete.
45. Show that $\{(x_n) \in l^2 : |x_n| < \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$ is a convex set with empty interior.
46. Suppose that $x \in l^p$ for some $p \geq 1$. Show that $\liminf_{p \rightarrow \infty} \|x\|_p \geq \|x\|_\infty$. Prove/disprove that $\lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty$.
47. Let M be a subspace of a normed linear space X . Then show that M is closed if and only if $\{y \in M : \|y\| \leq 1\}$ is closed in X .
48. Let $T : (C[0, \frac{\pi}{2}], \| \cdot \|_\infty) \rightarrow (C[0, \frac{\pi}{2}], \| \cdot \|_\infty)$ be defined by $(Tf)(x) = \int_{s=0}^x f(s) \sin s ds$. Show that T is not a contraction but T^2 is a contraction.
49. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous and let there exist $\alpha > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{y})\| \geq \alpha \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that $f(\mathbb{R}^n)$ is complete.
50. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a contraction and let $g(\mathbf{x}) = \mathbf{x} - f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-one and onto. Also, show that both g and $g^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous.
51. Find a neighborhood of $x = 0$ in which initial value problem $y' = \frac{x}{1+y^2}$ with $y(0) = 0$ has a unique solution.