# Assignment 1

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) There exists an unbounded subset $A$ of $\mathbb{R}$ such that $m^*(A) = 5$.
   (b) There exists an open subset $A$ of $\mathbb{R}$ such that $[\frac{1}{2}, \frac{3}{4}] \subset A$ and $m^*(A) = \frac{1}{4}$.
   (c) There exists an open subset $A$ of $\mathbb{R}$ such that $m^*(A) < \frac{1}{5}$ but $A \cap (a, b) \neq \emptyset$ for all $a, b \in \mathbb{R}$ with $a < b$.
   (d) If $A$ and $B$ are open subsets of $\mathbb{R}$ such that $A \subsetneq B$, then it is necessary that $m^*(A) < m^*(B)$.
   (e) There is a Lebesgue measurable set $A \subset \mathbb{R}$ such that $m(A) = 0$ but $m(\text{boundary}(A)) = \infty$.
   (f) If $A$ and $A \cup B$ are Lebesgue measurable subsets of $\mathbb{R}$, then $B$ is necessarily Lebesgue measurable subset of $\mathbb{R}$.
   (g) There exists a non-zero finite measure $\mu$ on $M(\mathbb{R})$, which is constant on every bounded open interval $(a, b)$ with $a < b$.

2. Examine whether $\mathcal{A}$ is a $\sigma$-algebra of subsets of $\mathbb{R}$, where
   (a) $\mathcal{A} = \{A \subset \mathbb{R} : m^*(A) = 0 \text{ or } m^*(\mathbb{R} \setminus A) = 0\}$.
   (b) $\mathcal{A} = \{A \subset \mathbb{R} : m^*(A) < +\infty \text{ or } m^*(\mathbb{R} \setminus A) < +\infty\}$.
   (c) $\mathcal{A} = \{A \subset \mathbb{R} : A \text{ or } \mathbb{R} \setminus A \text{ is an open subset of } \mathbb{R}\}$.

3. Let $X$ be an uncountable set. Show that the class $\{\{x\} : x \in X\}$ generates the $\sigma$-algebra $\{A \subset X : A \text{ is countable or } X \setminus A \text{ is countable}\}$.

4. Let $\mathcal{S}$ be a class of subsets of a nonempty set $X$ and let $A \subset X$. Show that $\sigma(\mathcal{S} \cap A) = \sigma(\mathcal{S}) \cap A$, where for each class $\mathcal{C}$ of subsets of $X$, $\mathcal{C} \cap A = \{C \cap A : C \in \mathcal{C}\}$.

5. Let $X$, $Y$ be nonempty sets and let $f : X \to Y$. If $\mathcal{S}$ is a class of subsets of $Y$, then show that $\sigma(f^{-1}(\mathcal{S})) = f^{-1}(\sigma(\mathcal{S}))$, where for each class $\mathcal{C}$ of subsets of $Y$, $f^{-1}(\mathcal{C}) = \{f^{-1}(C) : C \in \mathcal{C}\}$.

6. If $\mathcal{S}$ is a class of subsets of a nonempty set $X$ and if $A \in \sigma(\mathcal{S})$, then show that there exists a countable subclass $\mathcal{S}_0$ of $\mathcal{S}$ such that $A \in \sigma(\mathcal{S}_0)$.

7. Prove that every infinite $\sigma$-algebra on an infinite set is uncountable.

8. Show that $\mathcal{B}(\mathbb{R})$ is generated by each of the following classes.
   (a) $\{(a, +\infty) : a \in \mathbb{R}\}$
   (b) $\{(-\infty, a] : a \in \mathbb{Q}\}$
   (c) $\{[a, b) : a, b \in \mathbb{Q}, a < b\}$
   (d) $\{A \subset \mathbb{R} : A \text{ is compact}\}$

9. Let $E$ be a Borel subset of $\mathbb{R}$ and let $x \in \mathbb{R}$. Show that $x + E$ is a Borel subset of $\mathbb{R}$.

10. Examine whether $\mu^*$ is an outer measure on $\mathbb{R}$, where for each $A \subset \mathbb{R}$,
   (a) $\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is bounded,} \\ 1 & \text{if } A \text{ is unbounded.} \end{cases}$
   (b) $\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \text{ is nonempty and bounded,} \\ +\infty & \text{if } A \text{ is unbounded.} \end{cases}$

11. Consider the outer measure $\mu^*$ on $\mathbb{R}$, where for each $A \subset \mathbb{R}$, $\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ 1 & \text{if } A \text{ is uncountable.} \end{cases}$
   Determine all the $\mu^*$-measurable subsets of $\mathbb{R}$.

12. If $\mathcal{S} = \{\emptyset, [1, 2]\}$ and if $\mu(\emptyset) = 0$, $\mu([1, 2]) = 1$, then determine the outer measure $\mu^*$ on $\mathbb{R}$ induced by the set function $\mu : \mathcal{S} \to [0, +\infty)$.
   Also, determine all the $\mu^*$-measurable subsets of $\mathbb{R}$.

13. Let $(X, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \to [0, +\infty]$ be finitely additive with $\mu(\emptyset) = 0$. Show that $\mu$ is a measure on $\mathcal{A}$ if either of the following conditions is satisfied.

(a) For every increasing sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathcal{A}$, $\lim\limits_{n\to\infty} \mu(A_n) = \mu(\bigcup\limits_{n=1}^{\infty} A_n)$.

(b) For every decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathcal{A}$ with $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$, $\lim\limits_{n\to\infty} \mu(A_n) = 0$.

14. Let $\mu^*$ be an outer measure generated by a finite premeasure $\mu_o$ on an algebra $\mathcal{A}$ on a nonempty set $X$. Show that $E \subseteq X$ is $\mu^*$-measurable iff for each $\varepsilon > 0$, there exists a $\mu^*$-measurable set $G$ containing $E$ such that $\mu^*(G \setminus F) < \varepsilon$.

15. Let $f : [0, 2) \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1, \\ 3 - x & \text{if } 1 < x < 2. \end{cases}$
Find $m^*(A)$, where $A = f^{-1}((\frac{9}{16}, \frac{5}{4})) = \{x \in [0, 2) : f(x) \in (\frac{9}{16}, \frac{5}{4})\}$.

16. Let $B \subset A \subset \mathbb{R}$ such that $m^*(B) = 0$. Show that $m^*(A \setminus B) = m^*(A)$.

17. Let $A \subset \mathbb{R}$ such that $m^*(A) > 0$. Show that there exists $B \subset A$ such that $B$ is bounded and $m^*(B) > 0$.

18. If $G$ is a nonempty open subset of $\mathbb{R}$, then show that $m^*(G) > 0$.

19. Let $A$ be a countable subset of $\mathbb{R}$ and let $B \subset \mathbb{R}$ such that $m^*(B) = 0$. Show that $m^*(A+B) = 0$.

20. Prove or disprove: A subset $E$ of $\mathbb{R}$ is Lebesgue measurable iff $m^*(A \cup B) = m^*(A) + m^*(B)$ for each $A \subset E$ and for each $B \subset \mathbb{R} \setminus E$.

21. Let $E = \{x \in [0, 1] :$ The decimal representation of $x$ does not contain the digit $5\}$. Show that $m(E) = 0$.

22. Let $A_n \subset \mathbb{R}$ for $n \in \mathbb{N}$ such that $\sum\limits_{n=1}^{\infty} m^*(A_n) < \infty$. If $E = \{x \in \mathbb{R} : x \in A_n$ for infinitely many $n\}$, then show that $m(E) = 0$.

23. Show that a subset $E$ of $\mathbb{R}$ is Lebesgue measurable iff $m^*(I) = m^*(I \cap E) + m^*(I \setminus E)$ for every bounded open interval $I$ of $\mathbb{R}$.

24. Let $A \subset E \subset B \subset \mathbb{R}$ such that $A$, $B$ are Lebesgue measurable and $m(A) = m(B) < \infty$. Show that $E$ is Lebesgue measurable.
More generally, let $A \subset B \subset \mathbb{R}$ such that $A$ is Lebesgue measurable and $m^*(B) = m(A) < \infty$. Show that $B$ is Lebesgue measurable.

25. Let $A, B \subset \mathbb{R}$ such that $m^*(A) = 0$ and $A \cup B$ is Lebesgue measurable. Show that $B$ is Lebesgue measurable.

26. Let $A, B \subset \mathbb{R}$ such that $A$ is Lebesgue measurable and $m^*(A \triangle B) = 0$. Show that $B$ is Lebesgue measurable.

27. Let $A, B \subset \mathbb{R}$ be such that $A \cap B$ is Lebesgue measurable and $m^*(A \triangle B) = 0$. Show that $A$ and $B$ are Lebesgue measurable and $m(A) = m(B)$.

28. Let $A \subset \mathbb{R}$ such that $A \cap B$ is Lebesgue measurable for every bounded subset $B$ of $\mathbb{R}$. Show that $A$ is Lebesgue measurable.

29. Let $A$ and $B$ be subsets of $[0, 1]$ which satisfy $m^*(A \cup B) = m^*(A) + m^*(B)$. If $A \triangle B$ is Lebesgure measurable then prove that $A$ and $B$ are Lebesgure measurable.

30. Let $E$ be a Lebesgue measurable subset of $\mathbb{R}$ and let $A \subset \mathbb{R}$. Show that $m^*(E \cap A) + m^*(E \cup A) = m^*(E) + m^*(A)$.

31. Let $I$ and $J$ be disjoint open intervals in $\mathbb{R}$ and let $A \subset I$, $B \subset J$. Show that $m^*(A \cup B) = m^*(A) + m^*(B)$.

32. Let $A$ be a subset of $\mathbb{R}$ with $0 < m^*(A) < \infty$. Show that for each $\epsilon > 0$ there exist an open set $O$ containing $A$ and a compact set $K \subset \mathbb{R}$ such that $m(O \setminus K) < \epsilon$.