Assignment 1

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) $\{x \in \ell^1 : \|x\|_2 \leq 1\}$ is a bounded set in the Banach space $(\ell^1, \|\cdot\|_1)$.
   (b) $\{x \in \ell^1 : \|x\|_1 < 1\}$ is an open set in the normed linear space $(\ell^1, \|\cdot\|_2)$.
   (c) $\{x \in C[0,1] : \|x\|_\infty < 1\}$ is an open set in the normed linear space $(C[0,1], \|\cdot\|_1)$.
   (d) If (x_n) is a sequence in a Banach space X such that $\sum\limits_{n=1}^{\infty} n^2\|x_n\|^2 < \infty$, then the series $\sum\limits_{n=1}^{\infty} x_n$ must be convergent in X.

2. Examine whether $\|\cdot\|$ is a norm on $\mathbb{R}^n$, where
   (a) $\|(x_1, ..., x_n)\| = \left(\sum\limits_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$ for all $(x_1, ..., x_n) \in \mathbb{R}^n$ ($n \geq 2$ and $0 < p < 1$).
   (b) $\|(x_1, ..., x_n)\| = (x_1^2 + \frac{1}{2}x_2^2 + \cdots + \frac{1}{n}x_n^2)^{\frac{1}{2}}$ for all $(x_1, ..., x_n) \in \mathbb{R}^n$.

3. Examine whether $\|\cdot\|$ is a norm on $C[0,1]$, where $\|x\| = \min\{\|x\|_\infty, 2\|x\|_1\}$ for all $x \in C[0,1]$.

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function which vanishes at most at one point in $\mathbb{R}^n$ and satisfies $f(\alpha x) = |\alpha| f(x)$, for each $(\alpha, x) \in \mathbb{R} \times \mathbb{R}^n$. Show that f is a norm on $\mathbb{R}^n$.

5. Let X be a normed linear space and let $x \in X$. Show that $\|x\| = \inf\left\{\frac{1}{|\alpha|} : \alpha \in \mathbb{K} \setminus \{0\},\ \|\alpha x\| \leq 1\right\}$.

6. Let M be a proper closed subspace of a normed linear space X. Define a map $f : X \to \mathbb{R}$ by $f(x) = \inf\limits_{m \in M} \|x + m\|$. Show that f is an uniformly continuous function on X.

7. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $1 \leq p < q \leq \infty$, prove that
$$\|x\|_q \leq \|x\|_p \leq n^{\left(\frac{1}{p} - \frac{1}{q}\right)} \|x\|_q.$$

8. Let $\mathbb{Q} = \{r_1, r_2, \ldots,\}$ be an enumeration set of rational numbers. Define a sequence of functions $f_n : [0,1] \to \mathbb{R}$ by
$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \cap [0,1], \\ 0, & \text{otherwise.} \end{cases}$$
   Show that (f_n) is a Cauchy sequence in $(\mathcal{R}[0,1], \|\ .\ \|_1)$ but it does not converge to a function in $\mathcal{R}[0,1]$. Does (f_n) converge in $L^1[0,1]$?

9. Let $C^1[0,1]$ denote the space of all continuously differentiable functions on $[0,1]$. For $f \in C^1[0,1]$, define $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Show that space $(C^1[0,1], \|\ .\ \|)$ is a Banach space.

10. Let $X = \{f \in C^1[0,1] :\ f(0) = 0\}$. For $f \in X$, define $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$. Prove that $\|f\|_1 \leq 2\|f'\|_\infty$.

11. Let $X = \{f \in C^1[0,1] :\ f(0) = 0\}$. Then $\|f\| = \left(\int_0^1 |f'|^2\right)^{\frac{1}{2}}$ defines a norm on $C^1[0,1]$. Whether $(X,\ \|\ .\ \|)$ is a Banach space ?

12. Show that $L^p[0,1]$ is proper dense subspace of $L^1[0,1]$, whenever $1 < p < \infty$.

13. Let C_o be the class of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $|f(x)| < \epsilon$, for all $x \in \mathbb{R} \setminus K$. Show that $(C_o, \ \| \, . \, \|_\infty)$ is a Banach space.

14. Let $C_c(\mathbb{R}^n)$ denotes the class of all compactly supported continuous functions on $\mathbb{R}^n$.
    (i): Prove that $C_c(\mathbb{R})$ is a proper dense subspace of $L^p(\mathbb{R})$, whenever $1 \leq p < \infty$.
    (ii): Whether $C_c(\mathbb{R})$ is a dense subspace of $L^\infty(\mathbb{R})$?
    (iii): Prove that $C_c(\mathbb{R})$ is a dense subspace of $(C_o, \ \| \, . \, \|_\infty)$.
    (iii): Prove that $L^1 \cap L^p(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, whenever $1 < p < \infty$.

15. Let $1 \leq p < q < \infty$. Prove that $L^q[0,1]$ is a dense proper subspace of $L^p[0,1]$.

16. Let $S(\mathbb{R})$ be the space of simple functions on $\mathbb{R}$. Prove that $S(\mathbb{R})$ is dense in $\text{Ł}^p(\mathbb{R})$, for $1 \leq p < \infty$. Why $S(\mathbb{R})$ is not dense in $\text{Ł}^\infty(\mathbb{R})$?

17. Let (x_n) be a sequence in a normed linear space X which converges to a non-zero vector $x \in X$. Show that
$$\frac{x_1 + \cdots + x_n}{n^\alpha} \to x$$
if and only if $\alpha = 1$. If the sequence $x_n \to 0$, prove that
$$\frac{x_1 + \cdots + x_n}{n^\alpha} \to 0, \text{ for all } \alpha \geq 1.$$

18. Let M be a subspace of a normed linear space X. Then show that M is closed if and only if $\{y \in M : \ \|y\| \leq 1\}$ is closed in X.

19. Let $D = \{z \in \mathbb{C} : \ |z| < 1\}$. Let X be the class of all functions f which are analytic on D and continuous on $\bar{D}$. Define $\|f\|_\infty = \sup\{|f(e^{it})| : \ 0 \leq t \leq 2\pi\}$. Prove that $(X, \ \| \, . \, \|_\infty)$ is a Banach space.

20. Let M be a closed subspace of a normed linear space X. Prove that projection $\pi : \ X \to X/M$ defined by $\pi(x) = \tilde{x}$ is a continuous map.

21. Prove that $L^p(\mathbb{R})$ is separable for $1 \leq p < \infty$ but $L^\infty(\mathbb{R})$ is not separable.

22. Let M be a closed subspace of a normed linear space X. Then show that X is separable if and only if M and X/M both are separable.

23. For each $n \in \mathbb{N}$, let $x_n = 1 + \frac{(-1)^n}{n}$. Determine all p with $1 \leq p \leq \infty$ for which $(x_n) \in \ell^p$.

24. Let $1 \leq p < q \leq \infty$. Show that
    (a) $\ell^p \subsetneq c_0$
    (b) $\ell^p \subsetneq \ell^q$
    (c) $\|x\|_q \leq \|x\|_p$ for all $x \in \ell^p$

25. Can you find an element in c_0 which does not belong to ℓ^p for any $1 \le p < \infty$?

26. Prove the following:
(a) If $x \in \mathbb{R}^n$, then $\lim\limits_{p\to\infty} \|x\|_p = \|x\|_\infty$.
(b) If $x \in \ell^q$ for some $1 \le q < \infty$, then $\lim\limits_{p\to\infty} \|x\|_p = \|x\|_\infty$.
(c) If $x \in C[a,b]$, then $\lim\limits_{p\to\infty} \|x\|_p = \|x\|_\infty$.

27. Prove that c_{00} is dense in $(c_0, \|\cdot\|_\infty)$ and also in $(\ell^p, \|\cdot\|_p)$ for $1 \le p < \infty$.

28. Show that $\left\{(x_n) \in c_{00} : \sum\limits_{n=1}^{\infty} x_n = 0\right\}$ is dense in $(c_0, \|\cdot\|_\infty)$.

29. Show that $\{(x_n) \in \ell^2 : |x_n| \le \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$ is a compact convex subset of ℓ^2 with empty interior.

30. Let X be a metric space. Prove that the normed linear space $(C_b(X), \|\cdot\|_\infty)$ is finite dimensional iff X is finite.

31. Let X be a nonzero normed linear space, let $x, y \in X$ and let $\varepsilon, \delta > 0$. If $B_\varepsilon[x] = B_\delta[y]$, then show that $x = y$ and $\varepsilon = \delta$. Does the result remain true if X is assumed to be only a metric space?

32. Let Y be a subspace of a normed linear space X. Show that
(a) $Y^0 \ne \emptyset$ iff $Y = X$.
(b) Y is nowhere dense in X iff Y is not dense in X.

33. Let X be a nonzero normed linear space. Show that $\{x \in X : 1 < \|x\| \le 2\}$ is neither an open set nor a closed set in X.

34. Let A be a subset of a normed linear space. Show that $\overline{A} = \bigcap\limits_{r>0} (A + B_r(0))$.

35. Let A be a nonempty subset of a nonzero normed linear space X. Show that $\overline{A + B_X} = \{x \in X : d(x, A) \le 1\}$.

36. Show that $\|(\alpha_n) + c_0\| = \limsup\limits_{n\to\infty} |\alpha_n|$ for each $(\alpha_n) \in \ell^\infty$.

37. If K is a compact set in the Banach space $(\ell^\infty, \|\cdot\|_\infty)$, then show that $\{x + c_0 : x \in K\}$ is a compact set in the Banach space ℓ^∞/c_0 with the quotient norm.

38. Show that a normed linear space X is separable iff S_X is separable.

39. Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space. Show that there is a norm $\|\cdot\|_0$ on X such that $(X, \|\cdot\|_0)$ is nonseparable.

40. Let (x_n) be a sequence in a Banach space X. Which of the following conditions ensure(s) that (x_n) is convergent in X?
(a) $\|x_n - x_{n+1}\| \to 0$.

(b) $\sum\limits_{n=1}^{\infty} \|x_n - x_{n+1}\| < \infty$.

41. Let Y be a subspace of a normed linear space X. If for every sequence (y_n) in Y and for every $x \in X$ satisfying $\sum\limits_{n=1}^{\infty} y_n = x$, we have $x \in Y$, then show that Y is closed in X.

42. If a normed linear space X has a complete subspace Y such that X/Y is complete, then show that X is a Banach space.

43. If $1 \leq p < q \leq \infty$, then examine whether $(\ell^p, \|\cdot\|_q)$ is a Banach space.

44. Show that $(c_0, \|\cdot\|)$ is not a Banach space, where $\|(x_n)\| = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}|x_n|$ for all $(x_n) \in c_0$.

45. Consider the normed linear space $(\ell^\infty, \|\cdot\|)$, where $\|(x_n)\| = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}|x_n|$ for all $(x_n) \in \ell^\infty$. Examine whether $(\ell^\infty, \|\cdot\|)$ is a Banach space.

46. If $\|(x_n)\| = \sup\limits_{n\in\mathbb{N}} \left|\sum\limits_{i=1}^{n} x_i\right|$ for all $(x_n) \in \ell^1$, then show that $\|\cdot\|$ is a norm on ℓ^1.
Also, examine whether the normed linear space $(\ell^1, \|\cdot\|)$ is a Banach space.

47. Let $X = \left\{x \in C[0,1] : \int\limits_0^1 x(t)\,dt = 0\right\}$. Prove that $(X, \|\cdot\|_\infty)$ is an infinite dimensional Banach space.

48. Consider the normed linear space $(C[0,1], \|\cdot\|)$, where $\|x\| = \sup\{t|x(t)| : t \in [0,1]\}$. Show that $(C[0,1], \|\cdot\|)$ is not a Banach space.

49. If a normed linear space X has a complete subspace Y such that X/Y is complete, then show that X is a Banach space.

50. Let $(X, \|\cdot\|)$ be a normed linear space and let p be a seminorm on X. Show that $p : (X, \|\cdot\|) \to \mathbb{R}$ is continuous iff there exists $\alpha > 0$ such that $p(x) \leq \alpha\|x\|$ for all $x \in X$.

51. Let $(X, \|\cdot\|)$ be a normed linear space and let p be a seminorm on X. Show that $p : (X, \|\cdot\|) \to \mathbb{R}$ is continuous iff $\{x \in X : p(x) = 1\}$ is closed in $(X, \|\cdot\|)$.