

MA224: Real Analysis

(Assignment 2: Metric and Normed Linear Spaces)

January - April, 2026

1. State TRUE or FALSE giving proper justification for each of the following statements.
 - (a) There exists a metric space having exactly 36 open sets.
 - (b) It is impossible to define a metric d on \mathbb{R} such that only finitely many subsets of \mathbb{R} are open in (\mathbb{R}, d) .
 - (c) If A and B are open (closed) subsets of a normed vector space X , then $A + B = \{a + b : a \in A, b \in B\}$ is open (closed) in X .
 - (d) If A and B are closed subsets of $[0, \infty)$ (with the usual metric), then $A + B$ is closed in $[0, \infty)$.
 - (e) It is possible to define a metric d on \mathbb{R} such that the sequence $(1, 0, 1, 0, \dots)$ converges in (\mathbb{R}, d) .
 - (f) It is possible to define a metric d on \mathbb{R}^2 such that $((\frac{1}{n}, \frac{n}{n+1}))$ is not a Cauchy sequence in (\mathbb{R}^2, d) .
 - (g) It is possible to define a metric d on \mathbb{R}^2 such that in (\mathbb{R}^2, d) , the sequence $((\frac{1}{n}, 0))$ converges but the sequence $((\frac{1}{n}, \frac{1}{n}))$ does not converge.
 - (h) There exist two non-empty disjoint sets A and B in \mathbb{R} such that $\inf\{|x - y| : x \in A \text{ and } y \in B\} = 0$.
 - (i) If (x_n) is a sequence in a complete normed vector space X such that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$, then (x_n) must converge in X .
 - (j) If (f_n) is a sequence in $C[0, 1]$ such that $|f_{n+1}(x) - f_n(x)| \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$, then there must exist $f \in C[0, 1]$ such that $\int_0^1 |f_n(x) - f(x)| dx \rightarrow 0$ as $n \rightarrow \infty$.
 - (k) If (x_n) is a Cauchy sequence in a normed vector space, then $\lim_{n \rightarrow \infty} \|x_n\|$ must exist.
 - (l) $\{f \in C[0, 1] : \|f\|_1 \leq 1\}$ is a bounded subset of the normed vector space $(C[0, 1], \|\cdot\|_\infty)$.
 - (m) There exist a set $A \subset (\mathbb{R}, u)$ such that $\delta(A^o \cup \{0\}) = 0$ but $\delta((\bar{A})^o) = 1$, where δ stands for diameter.
 - (n) For $x, y \in l^\infty$, $d(x, y) = \min\{1, \limsup |x_n - y_n|\}$ define a metric on l^∞ .
 - (o) The sequence $f_n(t) = e^{-n^2 \sin \pi t}$ converge uniformly to 0 on $(0, 1)$.
2. Examine whether (X, d) is a metric space, where
 - (a) $X = \mathbb{R}$ and $d(x, y) = \frac{|x-y|}{1+|xy|}$ for all $x, y \in \mathbb{R}$.
 - (b) $X = \mathbb{R}$ and $d(x, y) = \min\{\sqrt{|x-y|}, |x-y|^2\}$ for all $x, y \in \mathbb{R}$.
 - (c) $X = \mathbb{R}$ and $d(x, y) = |x-y|^p$ for all $x, y \in \mathbb{R}$ ($0 < p < 1$).
 - (d) $X = \mathbb{R}$ and for all $x, y \in \mathbb{R}$, $d(x, y) = \begin{cases} 1 + |x-y| & \text{if exactly one of } x \text{ and } y \text{ is positive,} \\ |x-y| & \text{otherwise.} \end{cases}$
 - (e) $X = \mathbb{R}^2$ and $d(x, y) = (|x_1 - y_1| + |x_2 - y_2|^{\frac{1}{2}})^{\frac{1}{2}}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.
 - (f) $X = \mathbb{R}^n$ and $d(x, y) = [(x_1 - y_1)^2 + \frac{1}{2}(x_2 - y_2)^2 + \dots + \frac{1}{n}(x_n - y_n)^2]^{\frac{1}{2}}$ for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$.
 - (g) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} \min\{|z| + |w|, |z-1| + |w-1|\} & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$
 - (h) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} |z-w| & \text{if } \frac{z}{|z|} = \frac{w}{|w|}, \\ |z| + |w| & \text{otherwise.} \end{cases}$
 - (i) $X = \mathbb{C}$ and $d(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z, w \in \mathbb{C}$.
 - (j) $X =$ The class of all finite subsets of a nonempty set and $d(A, B) =$ The number of elements of the set $A \Delta B$ (the symmetric difference of A and B).

(k) $X = C[0, 1]$ and $d(f, g) = \left(\int_0^1 |f(t) - g(t)|^2 dt\right)^{\frac{1}{2}}$ for all $f, g \in C[0, 1]$.

3. Examine whether $\|\cdot\|$ is a norm on \mathbb{R}^2 , where for each $(x, y) \in \mathbb{R}^2$,

(a) $\|(x, y)\| = (\sqrt{|x|} + \sqrt{|y|})^2$.

(b) $\|(x, y)\| = \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}$.

(c) $\|(x, y)\| = \begin{cases} \sqrt{x^2 + y^2} & \text{if } xy \geq 0, \\ \max\{|x|, |y|\} & \text{if } xy < 0. \end{cases}$

4. Let $\|f\| = \min\{\|f\|_\infty, 2\|f\|_1\}$ for all $f \in C[0, 1]$. Prove that $\|\cdot\|$ is not a norm on $C[0, 1]$.

5. If $\mathbf{x} \in \mathbb{R}^n$, then show that $\lim_{p \rightarrow \infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$.

6. If $1 \leq p < q \leq \infty$, then show that $\|x\|_q \leq \|x\|_p$ for all $x \in \ell^p$.

7. Let d be a metric on a real vector space X satisfying the following two conditions:

(i) $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$,

(ii) $d(\alpha x, \alpha y) = |\alpha|d(x, y)$ for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$.

Show that there exists a norm $\|\cdot\|$ on X such that $d(x, y) = \|x - y\|$ for all $x, y \in X$.

8. Let \mathbb{R}^∞ be the real vector space of all sequences in \mathbb{R} , where addition and scalar multiplication are defined componentwise. Let $d((x_n), (y_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}$ for all $(x_n), (y_n) \in \mathbb{R}^\infty$. Show that d is a metric on \mathbb{R}^∞ but that no norm on \mathbb{R}^∞ induces d .

9. Let $(X, \|\cdot\|)$ be a nonzero normed vector space. Consider the metrics d_1, d_2 and d_3 on X :

$$\begin{aligned} d_1(x, y) &:= \min\{1, \|x - y\|\}, \\ d_2(x, y) &:= \frac{\|x - y\|}{1 + \|x - y\|}, \\ d_3(x, y) &:= \begin{cases} 1 + \|x - y\| & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \end{aligned}$$

for all $x, y \in X$. Prove that none of d_1, d_2 and d_3 is induced by any norm on X .

10. Let X be a normed vector space containing more than one point, let $x, y \in X$ and let $\varepsilon, \delta > 0$. If $B_\varepsilon[x] = B_\delta[y]$, show that $x = y$ and $\varepsilon = \delta$. Does the result remain true if X is assumed to be a metric space? Justify.

11. Examine whether the following sets are open/closed in \mathbb{R}^2 (with the usual metric).

(a) $\{(x, y) \in \mathbb{R}^2 : xy > 0\}$

(b) $\{(x, x) : x \in \mathbb{R}\}$

(c) $(0, 1) \times \{0\}$

(d) $\{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$

(e) $\{(x, y) \in \mathbb{R}^2 : x + y < 1\}$

(f) $\{(x, y) \in \mathbb{R}^2 : y \in \mathbb{Z}\}$

12. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is a closed/an open subset of \mathbb{R}^3 with respect to the usual metric on \mathbb{R}^3 .

13. Examine whether a finite subset of a metric space is open/closed.

14. For all $x, y \in \mathbb{R}$, let $d_1(x, y) = |x - y|$, $d_2(x, y) = \min\{1, |x - y|\}$ and $d_3(x, y) = \frac{|x - y|}{1 + |x - y|}$. If G is an open set in any one of the three metric spaces (\mathbb{R}, d_i) ($i = 1, 2, 3$), then show that G is also open in the other two metric spaces.

15. Let X be a nonzero normed vector space. Show that $\{x \in X : \|x\| < 1\}$ is not closed in X and $\{x \in X : \|x\| \leq 1\}$ is not open in X .

16. Show that $A = \{f \in C[0, 1] : \|f\|_1 < 1\}$ is an unbounded subset of the normed linear space $(C[0, 1], \|\cdot\|_\infty)$.

17. Let X be a normed vector space and let $Y(\neq X)$ be a subspace of X . Show that Y is not open in X .

18. Let F_n be a sequence of closed sets in \mathbb{R} such that $F_n \subset (n, n + 1]$ and $F_n \cap F_m = \emptyset$, whenever $m \neq n$. Show that $F = \bigcup_{n=1}^{\infty} F_n$ is a closed set in \mathbb{R} .

19. Let $f : (X, d) \rightarrow \mathbb{R}$ be a continuous function. Show that $\{x \in X : f(x) \neq 0\}$ is an open set in the metric space (X, d) .

20. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d) . Show that the sequence $(d(x_n, y_n))$ is convergent.

21. Let (x_n) be a sequence in a complete metric space (X, d) such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. Show that (x_n) converges in (X, d) .

22. Let (x_n) be a sequence in a metric space X such that each of the subsequences (x_{2n}) , (x_{2n-1}) and (x_{3n}) converges in X . Show that (x_n) converges in X .

23. Show that the following are incomplete metric spaces.

- (\mathbb{N}, d) , where $d(m, n) = \left| \frac{1}{m} - \frac{1}{n} \right|$ for all $m, n \in \mathbb{N}$
- $((0, \infty), d)$, where $d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|$ for all $x, y \in (0, \infty)$
- (\mathbb{R}, d) , where $d(x, y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|$ for all $x, y \in \mathbb{R}$
- (\mathbb{R}, d) , where $d(x, y) = |e^x - e^y|$ for all $x, y \in \mathbb{R}$

24. Examine whether the following metric spaces are complete.

- $([0, 1], d)$, where $d(x, y) = \left| \frac{x}{1-x} - \frac{y}{1-y} \right|$ for all $x, y \in [0, 1]$
- $((-1, 1), d)$, where $d(x, y) = \left| \tan \frac{\pi x}{2} - \tan \frac{\pi y}{2} \right|$ for all $x, y \in (-1, 1)$

25. For $X(\neq \emptyset) \subset \mathbb{R}$, let $d(x, y) = \frac{|x - y|}{1 + |x - y|}$ for all $x, y \in X$. Examine the completeness of the metric space (X, d) , where X is

- $[0, 1] \cap \mathbb{Q}$.
- $[-1, 0] \cup [1, \infty)$.
- $\{n^2 : n \in \mathbb{N}\}$.

26. For $f \in C^1[0, 1]$, define $\|f\| = \|f\|_1 + \|f\|_\infty$. Determine whether $(C^1[0, 1], \|\cdot\|)$ is a complete normed linear space.

27. Examine whether the sequence (f_n) is convergent in $(C[0, 1], d_\infty)$, where for all $n \in \mathbb{N}$ and for all $t \in [0, 1]$,

(a) $f_n(t) = \frac{nt^2}{1+nt}$.

(b) $f_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$.

(c) $f_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n}, \\ \frac{1}{nt} & \text{if } \frac{1}{n} < t \leq 1. \end{cases}$

(d) $f_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n}, \\ \frac{n}{n-1}(1-t) & \text{if } \frac{1}{n} < t \leq 1. \end{cases}$

28. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous and let there exist $\alpha > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{y})\| \geq \alpha \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that $f(\mathbb{R}^n)$ is complete.

29. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a contraction and let $g(\mathbf{x}) = \mathbf{x} - f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-one and onto. Also, show that both g and $g^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous.

30. Let $a_1 = 1$ and $a_{n+1} = \sqrt[3]{1 + a_n^2}$. Using fixed point theory, prove that the sequence $\{a_n\}$ is convergent, and that its limit satisfies the equation $x^3 - x^2 - 1 = 0$.

31. Using fixed point theory, determine all functions $f \in C[0, 1]$ such that $f(x) = \int_0^x (x-y) f(y) dy$.

32. For any $f \in C[0, 1]$, prove that

$$\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$