MA211(M): Real Analysis

(Assignment 1: Functions of several variables)

July - November, 2022

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) There exists a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\cos n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$.
   (b) There exists a non-constant continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = 5$ for all $(x, y) \in \mathbb{R}^2$ with $x^2 + y^2 < 1$.
   (c) There exists a one-one continuous function from $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ onto $\mathbb{R}^2$.
   (d) There exists a continuous function from $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ onto $\mathbb{R}^2$.
   (e) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that $f_x(0, 0)$ exists, then $f_y(0, 0)$ must exist.
   (f) There exists a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is differentiable only at $(1, 0)$.
   (g) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f_x(0, 0) = 0$. Then there exists some $\delta > 0$ such that $f(x, 0)$ is continuous on $(-\delta, \delta)$.
   (h) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that all the directional derivatives of f at $(0, 0)$ exist, then f must be differentiable at $(0, 0)$.
   (i) If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with $f(0, 0) = (1, 1)$ and $[f'(0, 0)] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then there cannot exist a differentiable function $g : \mathbb{R}^2 \to \mathbb{R}^2$ with $g(1, 1) = (0, 0)$ and $(f \circ g)(x, y) = (y, x)$ for all $(x, y) \in \mathbb{R}^2$.
   (j) A continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ cannot be one-one and onto if $\det[f'(x, y)] = 0$ for some $(x, y) \in \mathbb{R}^2$.
   (k) The equation $\sin(xyz) = z$ defines x implicitly as a differentiable function of y and z locally around the point $(x, y, z) = (\frac{\pi}{2}, 1, 1)$.
   (l) The equation $\sin(xyz) = z$ defines z implicitly as a differentiable function of x and y locally around the point $(x, y, z) = (\frac{\pi}{2}, 1, 1)$.

2. Let $\alpha \in (0, 1)$ and let $\mathbf{x}_n = (n^3\alpha^n, \frac{1}{n}[n\alpha])$ for all $n \in \mathbb{N}$. (For each $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x.) Examine whether the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$. Also, find $\lim\limits_{n\to\infty} \mathbf{x}_n$ if it exists.

3. Examine whether the following limits exist and find their values if they exist.

   $(a)\ \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^4 + y^2}$ $\quad (b)\ \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 - y^3}{x^2 + y^2}$ $\quad (c)\ \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2y^2 + (x^2 - y^2)^2}$

   $(d)\ \lim\limits_{(x,y)\to(0,0)} \dfrac{|x|}{y^2}e^{-|x|/y^2}$ $\quad (e)\ \lim\limits_{(x,y)\to(0,0)} \dfrac{1 - \cos(x^2 + y^2)}{(x^2 + y^2)^2}$ $\quad (f)\ \lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{x^2y^2 + 1} - 1}{x^2 + y^2}$

4. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0, 0)$, where for all $(x, y) \in \mathbb{R}^2$,
   (a) $f(x, y) = \begin{cases} xy & \text{if } xy \geq 0, \\ -xy & \text{if } xy < 0. \end{cases}$
   (b) $f(x, y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$
   (c) $f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

5. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, if for all $(x, y) \in \mathbb{R}^2$,
   (a) $f(x, y) = \begin{cases} \frac{xy}{x-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$

(b) $f(x,y) = \begin{cases} \frac{xy}{x^2-y^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{if } x^2 = y^2. \end{cases}$

(c) $f(x,y) = \begin{cases} xy & \text{if } xy \in \mathbb{Q}, \\ -xy & \text{if } xy \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

6. Let Ω be an open subset of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^m$ and $g : \Omega \to \mathbb{R}^m$ be continuous at $\mathbf{x}_0 \in \Omega$. If for each $\varepsilon > 0$, there exist $\mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}_0)$ such that $f(\mathbf{x}) = g(\mathbf{y})$, then show that $f(\mathbf{x}_0) = g(\mathbf{x}_0)$.

7. Let $A(\neq \emptyset) \subset \mathbb{R}^n$ be such that every continuous function $f : A \to \mathbb{R}$ is bounded. Show that A is a closed and bounded subset of $\mathbb{R}^n$.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous at $(x_0, y_0) \in \mathbb{R}^2$ and let $f(x_0, y_0) \neq 0$. Show that there exists $\delta > 0$ such that $f(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$ satisfying $(x - x_0)^2 + (y - y_0)^2 < \delta$.

9. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and let $f(\mathbf{x}) = T(\mathbf{x}) \cdot \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Find $f'(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$.

10. Examine the differentiability of f at $\mathbf{0}$, where
(a) $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $|f(\mathbf{x})| \leq \|\mathbf{x}\|_2^2$ for all $\mathbf{x} \in \mathbb{R}^n$.
(b) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \sqrt{|xy|}$ for all $(x,y) \in \mathbb{R}^2$.
(c) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = ||x| - |y|| - |x| - |y|$ for all $(x,y) \in \mathbb{R}^2$.
(d) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$
(e) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
(f) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \frac{\sin(x^2y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
(g) $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x,y) = \begin{cases} (\sin^2 x + x^2 \sin \frac{1}{x}, y^2) & \text{if } x \neq 0, \\ (0, y^2) & \text{if } x = 0. \end{cases}$
(h) $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$.
(i) $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|_2 \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

11. Let $f(x,y) = (x^2 + y^3, x^3 + y^2, 2x^2y^2)$ for all $(x,y) \in \mathbb{R}^2$. Examine whether $f : \mathbb{R}^2 \to \mathbb{R}^3$ is differentiable at $(1,2)$ and find $f'(1,2)$ if f is differentiable at $(1,2)$.

12. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, if for all $(x,y) \in \mathbb{R}^2$,
(a) $f(x,y) = \begin{cases} x^2 + y^2 & \text{if both } x, y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$
(b) $f(x,y) = \begin{cases} x^{4/3} \sin(\frac{y}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} (x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}}) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Show that f is differentiable at $(0,0)$ although neither f_x nor f_y is continuous at $(0,0)$.

14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \frac{x^2 y(x-y)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
Examine whether $f_{xy}(0, 0) = f_{yx}(0, 0)$.

15. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
Determine all the points of $\mathbb{R}^2$ where f_{xy} and f_{yx} are continuous.

16. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in \Omega$, let $f(\mathbf{x}_0) = 0$ and let $g : \Omega \to \mathbb{R}$ be continuous at $\mathbf{x}_0$. Prove that $fg : \Omega \to \mathbb{R}$, defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x} \in \Omega$, is differentiable at $\mathbf{x}_0$.

17. Let Ω be a nonempty open subset of $\mathbb{R}^n$ and let $g : \Omega \to \mathbb{R}^n$ be continuous at $\mathbf{x}_0 \in \Omega$. If $f : \Omega \to \mathbb{R}$ is such that $f(\mathbf{x}) - f(\mathbf{x}_0) = g(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0)$ for all $\mathbf{x} \in \Omega$, then show that f is differentiable at $\mathbf{x}_0$.

18. The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0, 0)$ in the directions of $(1, 2)$ and $(2, 1)$ are 1 and 2 respectively. Find $f_x(0, 0)$ and $f_y(0, 0)$.

19. Let $A \in GL(\mathbb{R}^n)$ and $\alpha \geq 2$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\|f(x)\| \leq k\|x\|^\alpha$, for some $k > 0$. Prove/disprove that the map $g = f + A$ is continuously differentiable at $\mathbf{0}$ and g is invertible in the neighborhood of $\mathbf{0}$.

20. Find all $\mathbf{v} \in \mathbb{R}^2$ for which the directional derivative $f'_{\mathbf{v}}(0, 0)$ exists, where for all $(x, y) \in \mathbb{R}^2$,
(a) $f(x, y) = \sqrt{|x^2 - y^2|}$.
(b) $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
(c) $f(x, y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$
(d) $f(x, y) = ||x| - |y|| - |x| - |y|$.

21. Let $f(x, y, z) = (x^3 y + y^2 z, xyz)$ and $g(x, y) = (x^2 y, xy, x - 2y, x^2 + 3y)$ for all $x, y, z \in \mathbb{R}$. Use chain rule to find $(g \circ f)'(\mathbf{a})$, where $\mathbf{a} = (1, 2, -3)$.

22. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable such that $f(1, 1) = 1$, $f_x(1, 1) = 2$ and $f_y(1, 1) = 5$. If $g(x) = f(x, f(x, x))$ for all $x \in \mathbb{R}$, determine $g'(1)$.

23. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = \varphi(x) + \psi(y)$ for all $(x, y) \in \mathbb{R}^2$, is differentiable.

24. Prove that a differentiable function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^m$ is homogeneous of degree $\alpha \in \mathbb{R}$ (*i.e.* $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and for all $t > 0$) iff $f'(\mathbf{x})(\mathbf{x}) = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

25. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is satisfying $f(rx) = r^{\frac{3}{2}} f(x)$ for all $(x, r) \in \mathbb{R}^n \times (0, \infty)$. Whether f is differentiable at $\mathbf{0}$?

26. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that $f_x(a, b) = f_y(a, b)$ for all $(a, b) \in \mathbb{R}^2$ and $f(a, 0) > 0$ for all $a \in \mathbb{R}$. Show that $f(a, b) > 0$ for all $(a, b) \in \mathbb{R}^2$.

27. Let Ω be an open subset of $\mathbb{R}^n$ such that $\mathbf{a}, \mathbf{b} \in \Omega$ and $S = \{(1-t)\mathbf{a} + t\mathbf{b} : t \in [0, 1]\} \subset \Omega$. If $f : \Omega \to \mathbb{R}^m$ is differentiable at each point of S, then show that there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(\mathbf{b}) - f(\mathbf{a}) = L(\mathbf{b} - \mathbf{a})$.

28. Let $f(x, y) = (2ye^{2x}, xe^y)$ for all $(x, y) \in \mathbb{R}^2$. Show that there exist open sets U and V in $\mathbb{R}^2$ containing $(0, 1)$ and $(2, 0)$ respectively such that $f : U \to V$ is one-one and onto.

29. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is locally invertible, if for all $(x, y) \in \mathbb{R}^2$,
    (a) $f(x, y) = (x^2 + y^2, xy)$.
    (b) $f(x, y) = (x^2 + xy + y^2, xy)$.
    (c) $f(x, y) = (\cos x + \cos y, \sin x + \sin y)$.

30. Determine all the points of $\mathbb{R}^3$ where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is locally invertible, if for all $(x, y, z) \in \mathbb{R}^3$,
    (a) $f(x, y, z) = (x + y, xy + z, y + z)$.
    (b) $f(x, y, z) = (x - xy, xy - xyz, xyz)$.
    (c) $f(x, y, z) = (x^2 + y^2, y^2 + z^2, z^2 + x^2)$.

31. Let $f(x, y) = (3x - y^2, 2x + y, xy + y^3)$ and $g(x, y) = (2ye^{2x}, xe^y)$ for all $(x, y) \in \mathbb{R}^2$. Examine whether $(f \circ g^{-1})'(2, 0)$ exists (with a meaningful interpretation of g^{-1}) and find $(f \circ g^{-1})'(2, 0)$ if it exists.

32. For $n \geq 2$, let $B = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < 1\}$ and let $f(\mathbf{x}) = \|\mathbf{x}\|_2^2 \mathbf{x}$ for all $\mathbf{x} \in B$. Show that $f : B \to B$ is differentiable and invertible but that $f^{-1} : B \to B$ is not differentiable at $\mathbf{0}$.

33. Using implicit function theorem, show that the system of equations

$$x^3(y^3 + z^3) = 0,$$
$$(x - y)^3 - z^2 = 7,$$

can be solved locally near the point $(1, -1, 1)$ for y and z as a differentiable function of x.

34. Show that the system of equations

$$x^2 - y\cos(uv) + z^2 = 0,$$
$$x^2 + y^2 - \sin(uv) + 2z^2 = 2,$$
$$xy - \sin u \cos v + z = 0,$$

implicitly defines (x, y, z) as a differentiable function of (u, v) near $x = 1$, $y = 1$, $z = 0$, $u = \frac{\pi}{2}$ and $v = 0$.

35. Using implicit function theorem, show that in a neighbourhood of any point $(x_0, y_0, u_0, v_0) \in \mathbb{R}^4$ which satisfies the equations

$$x - e^u \cos v = 0,$$
$$v - e^y \sin x = 0,$$

there exists a unique solution $(u, v) = \varphi(x, y)$ satisfying $\det[\varphi'(x, y)] = v/x$.

36. Show that in a neighbourhood of any point $(x_0, y_0, z_0) \in \mathbb{R}^3$ which satisfies the equations

$$x^4 + (x+z)y^3 - 3 = 0,$$
$$x^4 + (2x+3z)y^3 - 6 = 0,$$

there is a unique continuous solution $y = \varphi_1(x)$, $z = \varphi_2(x)$ of these equations.

37. Show that around the point $(0, 1, 1)$, the equation $xy - z\log y + e^{xz} = 1$ can be solved locally as $y = f(x, z)$ but cannot be solved locally as $z = g(x, y)$.

38. Show that the system of equations

$$3x + y - z + u^2 = 0,$$
$$x - y + 2z + u = 0,$$
$$2x + 2y - 3z + 2u = 0,$$

can be solved locally for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x, but not for x, y, z in terms of u.

39. Show that the system of equations

$$x^2 + y^2 - u^2 - v = 0,$$
$$x^2 + 2y^2 + 3u^2 + 4v^2 = 1,$$

defines (u, v) implicitly as a differentiable function of (x, y) locally around the point $(x, y, u, v) = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ but does not define (x, y) implicitly as a differentiable function of (u, v) locally around the same point.

40. Show that there are points $(x, y, z, u, v, w) \in \mathbb{R}^6$ which satisfy the equations

$$x^2 + u + e^v = 0,$$
$$y^2 + v + e^w = 0,$$
$$z^2 + w + e^u = 0.$$

Prove that in a neighbourhood of such a point there exist unique differentiable solutions $u = \varphi_1(x, y, z)$, $v = \varphi_2(x, y, z)$, $w = \varphi_3(x, y, z)$. If $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, find $\varphi'(x, y, z)$.

41. Find the 3rd order Taylor polynomial of $f(x, y, z) = x^2y + z$ about the point $(1, 2, 1)$.

42. Find the 4th order Taylor polynomial of $g(x, y) = e^{x-2y}/(1 + x^2 - y)$ about the point $(0, 0)$.

43. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Show that f is not one-one.