MA15010H: Multi-variable Calculus

(Assignment 1 Hint/ model solutions: Limits and continuity)
July - November, 2025

1. *Let $x, y \in \mathbb{R}^m$. Show that $\|x+y\| = \|x\| + \|y\|$ if and only if $y = 0$ or $x = \alpha y$ for some $\alpha \geq 0$.*

Solution. If $y = 0$, then $\|x+y\| = \|x\| = \|x\| + \|y\|$. Also, if $x = \alpha y$ for some $\alpha \geq 0$, then $\|x+y\| = \|(\alpha+1)y\| = (\alpha+1)\|y\|$ and $\|x\|+\|y\| = \alpha\|y\|+\|y\| = (\alpha+1)\|y\|$, so $\|x+y\| = \|x\|+\|y\|$.

Conversely, let $\|x+y\| = \|x\| + \|y\|$ and $y \neq 0$. Then,

$$\|x+y\|^2 = (\|x\| + \|y\|)^2.$$

This implies,

$$\|x\|^2 + 2x \cdot y + \|y\|^2 = \|x\|^2 + 2\|x\|\|y\| + \|y\|^2.$$

Thus $x \cdot y = \|x\|\|y\|$, and so $|x \cdot y| = \|x\|\|y\|$. By the equality condition in the Cauchy-Schwarz inequality, $x = \alpha y$ for some $\alpha \in \mathbb{R}$. Since we also have $x \cdot y = \|x\|\|y\|$, we obtain $\alpha y \cdot y = \|\alpha y\|\|y\|$, that is, $\alpha\|y\|^2 = |\alpha|\|y\|^2$. Since $\|y\| \neq 0$, we get $\alpha = |\alpha|$ and hence $\alpha \geq 0$. □

2. *Let $x, y \in \mathbb{R}^m$ and $r, s > 0$. Show that $B_r[x] \cap B_s[y] \neq \emptyset$ if and only if $\|x-y\| \leq r+s$.*

Solution. Suppose first that $B_r[x] \cap B_s[y] \neq \emptyset$. Then there exists $z \in B_r[x] \cap B_s[y]$, so

$$\|z-x\| \leq r \quad \text{and} \quad \|z-y\| \leq s.$$

By the triangle inequality,

$$\|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\| \leq r+s.$$

Conversely, assume that $\|x-y\| \leq r+s$. Define $z := \frac{s}{r+s}\,x + \frac{r}{r+s}\,y \in \mathbb{R}^m$. Then

$$\|z-x\| = \frac{r}{r+s}\|x-y\| \leq r \quad \text{and} \quad \|z-y\| = \frac{s}{r+s}\|x-y\| \leq s.$$

Hence $z \in B_r[x] \cap B_s[y]$. Therefore, $B_r[x] \cap B_s[y] \neq \emptyset$. □

3. *Let (x_n) be a sequence in $\mathbb{R}^m$. Show that (x_n) converges in $\mathbb{R}^m$ if and only if for each $x \in \mathbb{R}^m$, the sequence $(x_n \cdot x)$ converges in $\mathbb{R}$.*

Solution. Assume that (x_n) converges in $\mathbb{R}^m$ and let $x_0 \in \mathbb{R}^m$ such that $x_n \to x_0$. If $x \in \mathbb{R}^m$, then for all $n \in \mathbb{N}$,

$$|x_n \cdot x - x_0 \cdot x| = |(x_n - x_0) \cdot x| \leq \|x_n - x_0\|\|x\| \quad \text{(by Cauchy-Schwarz)}.$$

Since $x_n \to x_0$, we have $\|x_n - x_0\| \to 0$, and hence $|x_n \cdot x - x_0 \cdot x| \to 0$. Thus $x_n \cdot x \to x_0 \cdot x$ in $\mathbb{R}$. So the sequence $(x_n \cdot x)$ converges in $\mathbb{R}$.

Conversely, if $(x_n \cdot x)$ converges for all $x \in \mathbb{R}^m$, then in particular for $x = e_j$ $(1 \leq j \leq m)$, each coordinate sequence $x_n^{(j)} = x_n \cdot e_j$ converges in $\mathbb{R}$. Hence (x_n) converges in $\mathbb{R}^m$. □

4. *State **TRUE** or **FALSE** with justification for each statement.*

(i) If (x_n) is a sequence in $\mathbb{R}^m$ having no convergent subsequence, then it is necessary that $\lim_{n\to\infty} \|x_n\| = \infty$.

(ii) If (x_n, y_n) is a bounded sequence in $\mathbb{R}^2$ such that every convergent subsequence of (x_n, y_n) converges to $(0,1)$, then (x_n, y_n) must converge to $(0,1)$.

Solution.

(i) Let $r > 0$ and suppose, for contradiction, that $S = \{n \in \mathbb{N} : \|x_n\| \leq r\}$ is infinite. Then there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $\|x_{n_k}\| \leq r$ for all $k \in \mathbb{N}$. This subsequence (x_{n_k}) is bounded in $\mathbb{R}^m$ and by the Bolzano–Weierstrass theorem in $\mathbb{R}^m$, (x_{n_k}) has a convergent subsequence, which is a contradiction. Therefore, S is a finite set. Let $n_0 = 1$ if $S = \emptyset$ and $n_0 = \max S + 1$ if $S \neq \emptyset$. Then $\|x_n\| > r$ for all $n \geq n_0$. Thus, $\lim_{n\to\infty} \|x_n\| = \infty$. Statement is **TRUE**.

(ii) Suppose $(x_n, y_n) \not\to (0,1)$. Then there exists $\varepsilon > 0$ such that $(x_n, y_n) \notin B_\varepsilon((0,1))$ for infinitely many n, and hence we can find a strictly increasing sequence (n_k) such that $(x_{n_k}, y_{n_k}) \notin B_\varepsilon((0,1))$ for all k. Since (x_n, y_n) is bounded, so is this subsequence. By the Bolzano–Weierstrass theorem in $\mathbb{R}^2$, (x_{n_k}, y_{n_k}) has a convergent subsequence $(x_{n_{k_\ell}}, y_{n_{k_\ell}})$, which by the given converges to $(0,1)$. But this contradicts that $(x_{n_{k_\ell}}, y_{n_{k_\ell}}) \notin B_\varepsilon((0,1))$ for all ℓ. Hence, $(x_n, y_n) \to (0,1)$. Statement is **TRUE**.

□

5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$f(x,y) = \begin{cases} \frac{xy}{x^2-y^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{if } x^2 = y^2. \end{cases}$$

Determine all points of $\mathbb{R}^2$ where f is continuous.

Solution. Let $\varphi(x,y) = xy$ and $\psi(x,y) = x^2 - y^2$. As polynomial functions, $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ are continuous and $\psi(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$ with $x^2 \neq y^2$. Thus, f is continuous at each $(x,y) \in \mathbb{R}^2$ with $x^2 \neq y^2$.

If $(x,y) \in \mathbb{R}^2$ with $x^2 = y^2 \neq 0$, then for the sequence $(x + \frac{1}{n}, y) \to (x,y)$ but

$$\left| f\left(x + \frac{1}{n}, y\right) \right| = \left| \frac{(nx+1)y}{2x + \frac{1}{n}} \right| \to \infty.$$

Hence f is not continuous at (x,y).

Similarly, $(\frac{2}{n}, \frac{1}{n}) \to (0,0)$ but $f(\frac{2}{n}, \frac{1}{n}) = \frac{2}{3}$ for all n, hence $f(\frac{2}{n}, \frac{1}{n}) \not\to 0 = f(0,0)$. Therefore, the points of continuity of f are exactly $\{(x,y) \in \mathbb{R}^2 : x^2 \neq y^2\}$. □

6. *Let $\alpha, \beta > 0$, and define $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x,y) = \begin{cases} \frac{|x|^\alpha |y|^\beta}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is continuous if and only if $\alpha + \beta > 1$.

Solution. Suppose $\alpha + \beta > 1$, and let $(x_n, y_n) \to (0,0)$. Then $x_n \to 0$, $y_n \to 0$. For all n with $(x_n, y_n) \neq (0,0)$,

$$\begin{aligned} 0 \leq f(x_n, y_n) &\leq \frac{|x_n|^\alpha |y_n|^\beta}{\sqrt{x_n^2 + y_n^2}} \leq (x_n^2 + y_n^2)^{\alpha/2} (x_n^2 + y_n^2)^{\beta/2} (x_n^2 + y_n^2)^{-1/2} \\ &= (x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-1)} \end{aligned}$$

Since $\alpha + \beta > 1$, $(x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-1)} \to 0$, so $f(x_n, y_n) \to 0 = f(0,0)$. Thus, f is continuous at $(0,0)$ and is clearly continuous elsewhere.

Conversely, suppose f is continuous and $\alpha + \beta \leq 1$. Then $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{\sqrt{2}} n^{1-(\alpha+\beta)}.$$

If $\alpha + \beta = 1$, $f(\frac{1}{n}, \frac{1}{n}) \to \frac{1}{\sqrt{2}} \neq 0$; if $\alpha + \beta < 1$, the sequence is unbounded. This is a contradiction; thus $\alpha + \beta > 1$. □

7. *Let $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0, y_0) \in S$. Let $A = \{x \in \mathbb{R} : (x, y_0) \in S\}$ and $B = \{y \in \mathbb{R} : (x_0, y) \in S\}$. Define $\varphi(x) = f(x, y_0)$ for all $x \in A$ and $\psi(y) = f(x_0, y)$ for all $y \in B$. If f is continuous at (x_0, y_0), show $\varphi : A \to \mathbb{R}$ is continuous at x_0 and $\psi : B \to \mathbb{R}$ is continuous at y_0. Is the converse true? Justify.*

Solution. Let (x_n) be a sequence in A such that $x_n \to x_0$, and let (y_n) be a sequence in B such that $y_n \to y_0$. Then $(x_n, y_0), (x_0, y_n) \in S$ for all $n \in \mathbb{N}$ and

$$(x_n, y_0) \to (x_0, y_0), \quad (x_0, y_n) \to (x_0, y_0).$$

Since f is continuous at (x_0, y_0), $\varphi(x_n) = f(x_n, y_0) \to f(x_0, y_0) = \varphi(x_0)$, and $\psi(y_n) = f(x_0, y_n) \to f(x_0, y_0) = \psi(y_0)$. Therefore, φ is continuous at x_0 and ψ is continuous at y_0.

The converse is not true in general. For example, define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

Then f is not continuous at $(0, 0)$, because $(\frac{1}{n}, \frac{1}{n}) \to (0, 0)$ but $f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{2} \to \frac{1}{2} \neq 0$. However, $\varphi(x) = f(x, 0) = 0$ for all x, and $\psi(y) = f(0, y) = 0$ for all y. Thus, both φ and ψ are continuous at 0, but f is not continuous at $(0, 0)$. □

8. *If $S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 3\}$, determine (with justification) the interior S°.*

Solution. Let $(x_0, y_0) \in S$ and $0 < x_0 < 3$. Set $r = \min\{x_0, 3 - x_0\} > 0$. If $(x, y) \in B_r((x_0, y_0))$, then $|x - x_0| < r$, so $x_0 - r < x < x_0 + r$. Since $x_0 - r \geq 0$ and $x_0 + r \leq 3$, $x \in (0, 3)$. So $B_r((x_0, y_0)) \subset S$.

Suppose $(0, y) \in S^\circ$. Then there exists $r > 0$ such that $B_r((0, y)) \subset S$. But $(-\frac{r}{2}, y) \in B_r((0, y))$ and $-\frac{r}{2} < 0$, so $(-\frac{r}{2}, y) \notin S$. Contradiction. Similarly, for $(3, y) \in S^\circ$, $(3 + \frac{r}{2}, y) \notin S$. Thus,

$$S^\circ = \{(x, y) \in \mathbb{R}^2 : 0 < x < 3\}.$$

□