MA15010H: Multi-variable Calculus

(Assignment 1: Limits and continuity)

July - November, 2025

1. Let $x, y \in \mathbb{R}^m$. Show that $\|x+y\| = \|x\| + \|y\|$ iff $y = 0$ or $x = \alpha y$ for some $\alpha \geq 0$.

2. Let $x, y \in \mathbb{R}^m$ and $r, s > 0$. Show that $B_r[x] \cap B_s[y] \neq \emptyset$ iff $\|x - y\| \leq r + s$.

3. Let (x_n) be a sequence in $\mathbb{R}^m$. Show that (x_n) converges in $\mathbb{R}^m$ iff for each $x \in \mathbb{R}^m$, the sequence $(x_n \cdot x)$ converges in $\mathbb{R}$.

4. State TRUE or FALSE with justification for each statement:
   (a) If (x_n) in $\mathbb{R}^m$ has no convergent subsequence, then it is necessary that $\lim_{n\to\infty} \|x_n\| = \infty$.

   (b) If (x_n, y_n) is a bounded sequence in $\mathbb{R}^2$ and every convergent subsequence of (x_n, y_n) converges to $(0, 1)$, then (x_n, y_n) must converge to $(0, 1)$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \dfrac{xy}{x^2 - y^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{if } x^2 = y^2. \end{cases}$

   Determine all points in $\mathbb{R}^2$ where f is continuous.

6. Let $\alpha, \beta > 0$ and define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \begin{cases} \dfrac{|x|^\alpha |y|^\beta}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$ Show that f is continuous iff $\alpha + \beta > 1$.

7. Let $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$, and $(x_0, y_0) \in S$. Let $A = \{x \in \mathbb{R} : (x, y_0) \in S\}$, and $B = \{y \in \mathbb{R} : (x_0, y) \in S\}$. Define $\varphi(x) = f(x, y_0)$ for $x \in A$, and $\psi(y) = f(x_0, y)$ for $y \in B$. If f is continuous at (x_0, y_0), show $\varphi : A \to \mathbb{R}$ is continuous at x_0, and $\psi : B \to \mathbb{R}$ is continuous at y_0. Is the converse true? Justify.

8. If $S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 3\}$, determine (with justification) S^0.