Assignment 1

1. Let $X = C[0,1]$ be the space all the continuous functions on interval $[0,1]$. Prove that norms $\| \ . \ \|_\infty$ and $\| \ . \ \|_1$ on X are not equivalent.

2. Let $C^1[0,1]$ denote the space of all continuously differentiable functions on $[0,1]$. For $f \in C^1[0,1]$, define $\|f\| = \|f\|_\infty + \|f^{'}\|_\infty$. Show that space $(C^1[0,1], \| \ . \ \|)$ is a Banach space.

3. Does space $(C^1[0,1], \| \ . \ \|)$, where $\|f\| = \left(\int_0^1 |f|^2 + \int_0^1 |f^{'}|^2\right)^{\frac{1}{2}}$ a Banach space?

4. Let $f \in C^1[0,1]$. Write $\|f\| = \|f'\|_2 + \|f\|_\infty$. Whether $(C^1[0,1], \| \ . \ \|)$ is a Banach space?

5. Let $f \in C^1[0,1]$. Does $\|f\| = \min\left(\|f'\|_2, \ \|f\|_\infty\right)$ defines a norm on $C^1[0,1]$?

6. Let $X = \{f \in C^1[0,1] : \ f(0) = 0\}$. Then $\|f\| = \|f'\|$ is a norm on $C^1[0,1]$. Whether $(X, \ \| \ . \ \|)$ is a Banach space ?

7. Let $(V, \ \| \ . \ \|)$ be a normed linear space and X is the space of all the continuous functions from $[0,1]$ to V with $\|f\|_\infty = \sup_{t\in[0,1]} \|f(t)\|$. Prove that $(X, \ \| \ . \ \|_\infty)$ is a Banach space.

8. Let X be the class of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $|f(x)| < \epsilon$, for all $x \in \mathbb{R} \setminus K$. Show that $(X, \ \| \ . \ \|_\infty)$ is a Banach space.

9. Let $1 \leq p < \infty$. Let X_p be a class of all the Riemann integrable functions on $[0,1]$. Prove that $\|f\|_p = \left(\int_0^1 |f|^p\right)^{\frac{1}{p}} < \infty$. Prove that $(X_p, \ \| \ . \ \|_p)$ is a normed linear space but not complete.

10. Let $1 \leq p < \infty$. Let $L^p[0,1] = \{f : [0,1] \to \mathbb{C}, f \text{ is Lebesgue measurable } \}$ with $\|f\|_p = \left(\int_0^1 |f|^p\right)^{\frac{1}{p}} < \infty$. show that $L^p[0,1]$ is proper dense subspace of $L^1[0,1]$, whenever $1 < p < \infty$.

11. Let $1 \leq p < \infty$. Let $L^p(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C}, f \text{ is Lebesgue measurable } \}$ with $\|f\|_p = \left(\int_{\mathbb{C}} |f|^p\right)^{\frac{1}{p}} < \infty$. Let $C_c(\mathbb{C})$ be the class of all compactly supported functions on $\mathbb{C}$. Prove that $C_c(\mathbb{C})$ is proper dense subspace of $L^p(\mathbb{C})$, whenever $1 \leq p < \infty$. Whether $C_c(\mathbb{C})$ is a dense subspace of $L^\infty(\mathbb{C})$?

12. Let (x_n) be a sequence in a normed linear space X which converges to a non-zero vector $x \in X$. Show that
$$\frac{x_1 + \cdots + x_n}{n^\alpha} \to x$$
if and only if $\alpha = 1$. If the sequence $x_n \to 0$, prove that
$$\frac{x_1 + \cdots + x_n}{n^\alpha} \to 0, \text{ for all } \alpha \geq 1.$$

13. Prove that $l^\infty(\mathbb{N}) = \{x = (x_1, x_2, \ldots) : \ \|x\|_\infty = \sup_j |x_j|\}$ is a Banach space but not separable.

14. Let M be a subspace of a normed linear space X. Then show that M is closed if and only if $\{y \in M : \ \|y\| \leq 1\}$ is closed in X.

15. Let $D = \{z \in \mathbb{C} : \ |z| < 1\}$. Let X be the class of all functions f which are analytic on D and continuous on $\bar{D}$. Define $\|f\|_\infty = \sup\{|f(e^{it})| : \ 0 \leq t \leq 2\pi\}$. Prove that $(X, \ \| \, . \, \|_\infty)$ is a Banach space.

16. A normed linear space X is finite dimensional if and only if any ball $B_r(x)$ in X is compact.

17. Show that unit ball $\{x \in l^1(\mathbb{N}) : \ \|x\|_1 \leq 1\}$ is not compact in $l^1(\mathbb{N})$, without using the statement of Q16.

18. Show that unit ball $\{x \in l^\infty(\mathbb{N}) : \ \|x\|_\infty \leq 1\}$ is not compact $l^\infty(\mathbb{N})$, without using the statement of Q16.

19. Let M be a closed subspace of a normed linear space X. Prove that projection $\pi : \ X \to X/M$ defined by $\pi(x) = \tilde{x}$ is a continuous map.

20. Let X be a normed linear space. Prove that norm of any $x \in X$, can be expressed as $\|x\| = \inf\{|\alpha| : \ \alpha \in \mathbb{C} \setminus \{0\} \text{ with } \|x\| \leq |\alpha|\}$.

21. Let M be a closed subspace of a normed linear space X. Then show that X is separable if and only if M and X/M both are separable.

22. Does there exist a separable Banach space which has no Schauder basis ?

23. Let X be a separable Banach space. Prove that there exists a closed subspace M of $l^1(\mathbb{N})$ such that X is isomorphic to $l^1(\mathbb{N})/M$.