ADVANCED COURSE ON HARDY SPACES

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PRELIMINARY

Let us denote the unit circle in the complex plane by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Write $z = e^{i\theta}$; $0 \leq \theta < 2\pi$. Then $\mathbb{T} = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$. Consider $\varphi : \mathbb{R} \to \mathbb{T}$ defined by $\varphi(x) = e^{ix}$. Then φ is a group homomorphism with $\ker(\varphi) = 2\pi\mathbb{Z}$. Hence $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$. If $f : \mathbb{T} \to \mathbb{C}$, then f can be identified on $\mathbb{R}$ by $\tilde{f} : \mathbb{R} \to \mathbb{C}$ via the relations

$$\tilde{f}(x) = \tilde{f}(\tilde{x} + 2\pi k) = f(\tilde{x}),$$

where $k \in \mathbb{Z}$ and $\tilde{x} \in [0, 2\pi)$. That is, the function on $\mathbb{T}$ can be identified with 2π periodic functions on $\mathbb{R}$, which allow understanding the notions of continuity, Lebesgue integrability, etc. on the unit circle $\mathbb{T}$. Further, the arch length measure on $\mathbb{T}$ can be identified with the restriction of Lebesgue measure on $[0, 2\pi)$ in the following way.

Denote $dm = \frac{d\theta}{2\pi}$, where m can be realized by $m\{e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\} = \frac{\theta_2 - \theta_1}{2\pi}$ with $0 \leq \theta_2 - \theta_1 < 2\pi$. Here m is known as the normalized Lebesgue measure on $\mathbb{T} \cong [0, 2\pi)$. Hence if f is continuous on $\mathbb{T}$, then

$$\int_{\mathbb{T}} f(z)dz = \int_0^{2\pi} \tilde{f}(t)dm(t). \tag{0.1}$$

Now onwards, we shall identify function $\tilde{f}$ on $\mathbb{R}$ by f itself and $dm(t) = dt$. Moreover, m is translation invariant on $[0, 2\pi)$ and

$$\int_0^{2\pi} f(t - t_o)dt = \int_0^{2\pi} f(t)dt,$$

where $t_o \in [0, 2\pi)$.

0.1. **Complex Borel measure.** The Borel σ-algebra $\mathcal{B}(\mathbb{T})$ is the smallest σ-algebra generated by all open subsets (open arches) in $\mathbb{T}$, where every member of $\mathcal{B}(\mathbb{T})$ is known as a Borel set. For simplicity, we write $\mathcal{B}$ for $\mathcal{B}(\mathbb{T})$.

A function $f : \mathbb{T} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called Borel measurable if $f^{-1}(U) \in \mathcal{B}(\mathbb{T})$ for every open set U of one point compactification space $\hat{\mathbb{C}}$. Typically, U is either an open subset of $\mathbb{C}$ in its usual topology or $U = \hat{\mathbb{C}} \setminus K$, where K is a compact subset of $\mathbb{C}$.

A complex Borel measure on $\mathbb{T}$ is a set function $\mu : \mathcal{B}(\mathbb{T}) \to \mathbb{C}$ satisfying $\mu(\emptyset) = 0$ and

$$\mu(E) = \sum_{j=1}^{\infty} \mu(B_j) \tag{0.2}$$

for every countable partition $\{B_j\}_{n=1}^{\infty}$ of $E \in \mathcal{B}(\mathbb{T})$. It follows that the series in the right-side of (0.2) must be absolutely convergence unless μ is a non-negative measure. Thus, $|\mu(\mathbb{T})| < \infty$ necessarily satisfied if μ is not a non-negative measure. Consequently, μ satisfies

$$\mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} \mu(B_j)$$

for every disjoint sequence $\{B_j\}_{n=1}^{\infty}$ in $\mathcal{B}(\mathbb{T})$. We denote the space of all finite complex Borel measures by $\mathcal{M}(\mathcal{B})$. For $\mu \in \mathcal{M}(\mathcal{B})$, define

$$\|\mu\| = \sup\Big\{ \sum_{j=1}^{\infty} |\mu(B_j)| : \bigcup_{j=1}^{\infty} B_j = \mathbb{T} \Big\}.$$

The space $(\mathcal{M}(\mathcal{B}), \|\cdot\|)$ is a Banach space. Here $\|\cdot\|$ is known as the total variation norm, and $\|\mu\| = |\mu|(\mathbb{T})$.

Exercise 0.1. Show that

$$|\mu|(\mathbb{T}) = \sup\Big\{ \sum_{i=1}^{\infty} |\mu(B_i)| : \bigcup_{i=1}^{\infty} B_i = \mathbb{T} \Big\} = \sup\Big\{ \sum_{i=1}^{k} |\mu(B_i)| : \bigcup_{i=1}^{k} B_i = \mathbb{T} \Big\}.$$

(Hint: If $\{B_i\}_{i=1}^{\infty}$ is a countable cover of $\mathbb{T}$, then $\sum\limits_{i=1}^{\infty} |\mu(B_i)| < \infty$.)

For $\mu \in \mathcal{M}(\mathcal{B})$, define a linear functional T_μ on $C(\mathbb{T})$ by $T_\mu(f) = \int\limits_{\mathbb{T}} f d\mu$. Then $\|T_\mu\| = \sup\{|T_\mu(f)| : \|f\|_\infty \leq 1\} = \|\mu\|$. Thus, every $\mu \in \mathcal{M}(\mathcal{B})$ defines a bounded linear functional on $C(\mathbb{T})$ and vice-versa due to the following result.

Theorem 0.2. (Reisz representation theorem) *Let T be a bounded linear functional on $C(\mathbb{T})$, then there exists unique $\mu \in \mathcal{M}(\mathcal{B})$ such that $T = T_\mu$.*

1. Invariant subspaces of $L^2(\mu)$

In this section, consider shift-invariant subspaces of square integrable functions on $\mathbb{T}$. Let

$$L^2(\mathbb{T},\mu) = \{f : \mathbb{T} \to \mathbb{C} \text{ is measurable and} \|f\|_2^2 = \int_{\mathbb{T}} |f|^2 d\mu < \infty\},$$

where μ is a finite complex Borel measure on $\mathbb{T}$.

For $f \in L^1(\mathbb{T}, m)$, we define the Fourier coefficients of f by $\hat{f}(n) = \int\limits_0^{2\pi} e^{-int} f(t)dt$, where $n \in \mathbb{Z}$, and the corresponding Fourier series is $f \sim \sum\limits_{n=-\infty}^{\infty} e^{int}\hat{f}(n)$. Consider an operator S on $L^2(\mathbb{T}, m)$ defined by

$$S(f)(z) = zf(z), \tag{1.1}$$

where $z \in \mathbb{T}$. Then $\widehat{(Sf)}(n) = \hat{f}(n-1)$. That is, the Fourier coefficients got a right-shift due to the action of S. The operator S is known as the shift operator. The following question can be raised.

Question 1.1. What are the shift-invariant subspaces E of $L^2(\mathbb{T}, \mu)$?

That is, when $zE \subseteq E$? We shall use the notation clos$E$ for the closure of E, and $\bar{E}$, the complex conjugate of E. We always consider E to be a closed subspace unless it is specified.

Example 1.2. When $f \in L^2(\mu)$, the space $E_f = \overline{\text{span}}\{z^n f \,: n \geq 0\}$ is shift-invariant.

Further, what are $f \in L^2(\mu)$ such that $E_f = L^2(\mu)$? If so, we say f is a cyclic vector. More generally, we consider identifying $f \in L^2(\mu)$ such that $zE_f = E_f$.

Let E be a closed subspace of L^2. Typically, we discuss the characterization of the following two distinct cases.

We say E is **simply invariant** (or **1-invariant**) if $zE \subset E$ and $zE \neq E$. On the other hand, when $zE = E$, we say E is **doubly invariant** (or **2-invariant**). Note that $zE = E$ if and only if $\bar{z}E = E$ (since $z\bar{z} = |z|^2 = 1$). This means $zE \subseteq E$ and $\bar{z}E \subseteq E$, and hence E is known as reducing space as well.

For a measurable set $\sigma \subset \mathbb{T}$, the space $E_\sigma = \chi_\sigma L^2(\mu) = \{\chi_\sigma f : f \in L^2(\mu)\} = \{f \in L^2(\mu) : f = 0 \text{ a.e.}\mu \text{ on} \mathbb{T} \setminus \sigma\}$ satisfies $zE_\sigma = E_\sigma$.

Question 1.3. Does every reducing subspace look like E_σ?

Theorem 1.4. (Norbert Wiener) *Let $E \subset L^2(\mathbb{T}, \mu)$. Then $zE = E$ if and only if there exists a unique (up to set of measure zero) measurable set $\sigma \subset \mathbb{T}$ such that $E = \chi_\sigma L^2(\mu)$.*

Proof. Suppose $zE = E$. Let P_E be the orthogonal projection of $L^2(\mu)$ onto E. Set $\chi = P_E 1$ (the evaluation of P_E at the constant function 1). Then $\chi \in E$ and $1 - \chi = (I - P_E)1 \in E^\perp$. But $z^n E \subseteq E$, implies $z^n \chi \in E$ and hence $z^n \chi \perp 1 - \chi$, $\forall\, n \in \mathbb{Z}$. That is,

$$\int_{\mathbb{T}} z^n \chi (1 - \bar{\chi}) d\mu = 0, \forall\, n \in \mathbb{Z}. \tag{1.2}$$

Let $g = \chi(1 - \bar{\chi})$, then $d\nu = g d\mu$ is a finite complex Borel measure because of $\chi \in L^1(\mu)$. Thus, by (1.2), $T_\nu : L^2(\mu) \to \mathbb{C}$ defined by $T_\nu(f) = \int\limits_{\mathbb{T}} f d\nu$ satisfies $T_\nu(z^n) = 0$. Since trigonometric polynomials are dense in $C(\mathbb{T})$, it follows that $T_\nu(C(\mathbb{T})) = \{0\}$. By Riesz representation theorem, $T_\nu = 0$ and hence $\nu = 0$. (Note that $\|T_\nu\| = \|\nu\|$). That is, $g = \chi(1 - \bar{\chi}) = 0$. This implies that $\chi = |\chi|^2$. Thus, χ takes values either 0 or 1. Let $\sigma = \{t \in \mathbb{T} : \chi(t) = 1\}$. Then σ is measurable. For simplicity, let $\mathbb{P}$ denotes the space of all trigonometric polynomials on $\mathbb{T}$. Since $\chi \in E$, we get $z^n \chi \in E$ and hence $\chi \mathbb{P} \subset E$. This implies $\operatorname{clos}(\chi \mathbb{P}) \subseteq E$. On the other hand, $\operatorname{clos}(\chi \mathbb{P}) = \chi L^2(\mu)$, as we know $\operatorname{clos} \mathbb{P} = L^2(\mu)$. Thus, $\chi L^2(\mu) \subseteq E$. Therefore, it remains to show that $\chi L^2(\mu) = E$.

For this, let $f \in E$ and $f \perp z^n \chi$, $\forall\, n \in \mathbb{Z}$ (since $\operatorname{clos}(\chi \mathbb{P}) = \chi L^2(\mu)$). Since $z^n f \in E$ and $1 - \chi \perp z^n f$, $\forall\, n \in \mathbb{Z}$. It follows that

$$\int_{\mathbb{T}} f \bar{\chi} \bar{z}^n d\mu = \int_{\mathbb{T}} z^n f (1 - \bar{\chi}) d\mu = 0 \tag{1.3}$$

$\forall\, n \in \mathbb{Z}$. Thus, (1.3) is satisfied by every polynomial $p \in \mathbb{P}$, and hence for every function $g \in C(\mathbb{T})$ in place of p. By Theorem 0.2, we get $f\bar{\chi} = f(1 - \bar{\chi}) = 0$ a.e. μ. This implies that $f = 0$ a.e. μ. Thus, $\chi L^2(\mathbb{T}) = E$. $\square$

1.1. **Simply invariant subspaces of $L^2(\mu)$.** Let $\mathcal{B} = \{z^n\}_{n\in\mathbb{Z}}$. Notice that the Fourier series of $f \in L^2(\mathbb{T}, m)$ with respect to the orthonormal basis $\mathcal{B}$ is $f \sim \sum \hat{f}(n)z^n$, where $\hat{f}(n) = \int_{\mathbb{T}} f\bar{z}^n dm$. This implies that $L^2(\mathbb{T}, m)$ can be identified with $l^2(\mathbb{Z})$. Since $\widehat{(z^k f)}(n) = \hat{f}(n-k)$, multiplication operator $f \mapsto zf$ acts as a right-shift operator on $l^2(\mathbb{Z})$. And hence it is legitimate to consider the space

$$H^2 = \overline{\text{span}}\{z^n : n \geq 0\} = \{f \in L^2(m) : \hat{f}(n) = 0, n < 0\},$$

known as **Hardy space.** The space H^2 is a simply invariant subspace of $L^2(m)$, and plays a prominent role in complex and harmonic analysis H^2.

The following theorem says that all the simply invariant subspaces have a somewhat similar structure.

Theorem 1.5. (A. Beurling, H. Helson) *Let E be a closed subspace of $L^2(\mathbb{T})$ and $zE \subset E$, $zE \neq E$. Then there exists a unique Θ (up to constant of modulus 1) with $|\Theta| = 1$ a.e. m on $\mathbb{T}$ such that $E = \Theta H^2$.*

Notice that $f \mapsto \Theta f$ is an isometry on $L^2(m)$, and hence ΘH^2 is closed.

Proof. Since $zE \subsetneq E\,(zE \neq E)$, we consider the orthogonal complement of zE in E, and denote it by $E \ominus zE = (zE)^{\perp}$. Then $E \ominus zE$ is non-trivial, and consider $\Theta \in E \ominus zE$ with $\|\Theta\|_2 = 1$. Notice that $\Theta \in E$ and $\Theta \perp zE$. Hence $z^n\Theta \in zE, \forall\, n \geq 1$ and $\Theta \perp z^n\Theta, \forall\, n \geq 1$.

$$\int_0^{2\pi} \bar{\Theta}\Theta z^n dm = \int_0^{2\pi} |\Theta|^2 z^n dm = 0, \forall\, n \geq 1.$$

By taking complex conjugate, we have

$$\int_0^{2\pi} |\Theta|^2 \bar{z}^n dm = 0, \forall\, n \geq 1.$$

This implies that $\widehat{(|\Theta|^2)}(n) = 0, \forall\, n \in \mathbb{Z} \setminus \{0\}$. By the uniqueness of Fourier series, it follows that $|\Theta|^2 = c\,(\text{constant})$ a.e. m, and we get $1 = \int_0^{2\pi} |\Theta|^2 dm = c$. Thus, $|\Theta| = 1$ a.e. m. Clearly, $f \mapsto \Theta f$ is an isometry. Note that $\Theta \in E$. Hence $z^n\Theta \in E, \forall\, n \geq 0$, implies linear span of $\{z^n : n \geq 0\}$ has the same property. Let $\mathbb{P}_+ = \text{span}\{z^n : n \geq 0\}$.

Then $\Theta\,\mathbb{P}_+ \subset E$ and $\operatorname{clos}(\Theta\,\mathbb{P}_+) = \Theta\operatorname{clos}(\mathbb{P}_+) = \Theta H^2$. Thus, $\Theta H^2 \subseteq E$. It only remains to show that ΘH^2 coincides with E.

Let $f \in E$ and $f \perp \Theta H^2$. We claim that $f = 0$. Since $f \perp \Theta H^2$, we get $f \perp \Theta z^n$, $\forall\, n \geq 0$. Also, $f \in E$ implies $z^n f \in zE$, $\forall\, n \geq 1$ and hence $z^n f \perp \Theta$, $\forall\, n \geq 1$ since $\Theta \perp zE$. Thus,

$$\int_{\mathbb{T}} f\bar{\Theta}\bar{z}^n dm = 0,\ \forall\, n \geq 0 \text{ and } \int_{\mathbb{T}} z^n f\bar{\Theta} dm = 0,\ \forall\, n \geq 1.$$

That is, $\widehat{(f\bar{\Theta})}(n) = 0$, $\forall\, n \in \mathbb{Z}$. This implies $f\bar{\Theta} = 0$ a.e. m. Since $|\Theta| = 1$ a.e., we get $f = 0$ a.e. m.

Uniqueness: Let $\Theta_1 H^2 = \Theta_2 H^2$ and $|\Theta_1| = |\Theta_2| = 1$ a.e. on $\mathbb{T}$. Then $\Theta_1\bar{\Theta}_2 H^2 = H^2$ and we get $\Theta_1\bar{\Theta}_2 \in H^2$. Also, by symmetry $\Theta_2\bar{\Theta}_1 \in H^2$, or $\Theta_1\bar{\Theta}_2 \in \bar{H}^2$. But $H^2 \cap \bar{H}^2 =$ constant. (Hint: If $f \in H^2$, then $\hat{f}(n) = 0$, $n < 0$ and $\bar{f} \in H^2$, then $(\widehat{\bar{f}})(n) = \overline{\hat{f}(-n)} = 0$, $n < 0$. This means $\hat{f}(n) = 0, \forall\, n \in \mathbb{Z} \setminus \{0\}$.) Hence $\Theta_1\bar{\Theta}_2 = c$. Since $|\Theta_1||\bar{\Theta}_2| = 1$, we have $\Theta_1 = c\,\bar{\Theta}_2$, where $|c| = 1$. $\square$

Corollary 1.6. (Beurling theorem) *Let $E \neq \{0\}, E \subset H^2$ and $zE \subset E$. Then there exists $\Theta \in H^2$ with $|\Theta| = 1$ a.e. on $\mathbb{T}$ such that $E = \Theta H^2$.*

Proof. It is impossible that $\bar{z}E \subset E$. On the contrary, suppose this could be the case. Then for $f \in E$ with $f \neq 0$, there exists $n \in \mathbb{N}$ such that $\hat{f}(n) \neq 0$. By assumption, $\overline{z}^{n+1} f \in E$. However, $(\widehat{\overline{z}^{n+1} f})(-1) = \hat{f}(n) \neq 0$ implies $\bar{z}^{n+1} f \notin H^2$ leads to a contradiction. This means E is simply invariant, and in view of Theorem 1.5 (Beurling-Helson), it follows that $E = \Theta H^2$ and $\Theta \in H^2$ by definition of H^2. $\square$

Definition 1.7. A function $\Theta \in H^2$, with $|\Theta| = 1$ a.e. is called **inner** function.

1.2. **Uniqueness theorem in H^2.**

Theorem 1.8. *If $f \in H^2$ and $f = 0$ on a set of positive measure, then $f = 0$ a.e. on $\mathbb{T}$.*

Proof. For $f \neq 0, E_f = \overline{\text{span}}\{z^n f : n \geq 0\} \subset H^2$ and $zE_f \subset E_f = \Theta H^2$, where Θ is an inner function. Let $\sigma = \{z \in \mathbb{T} : f(z) = 0\}$, Then $m(\sigma) > 0$. Let us verify that $g|_\sigma = 0, \forall\, g \in E_f$. Since $g \in E_f$, there exists sequence $p_n \in \mathbb{P}_+$ (the space of all polynomials) such that $p_n f \to g$ in $L^2(m)$. Hence

$$0 \leq \int_\sigma |g|^2 dm = \int_\sigma |g - p_n f|^2 \leq ||g - p_n f||_2^2 \to 0 \text{ as} n \to \infty.$$

Implies $g|_\sigma = 0$ a.e. m. In particular, for $g = \Theta$, $\Theta|_\sigma = 0$, which is a contradiction. □

1.3. **Invariant subspaces of $L^2(\mu)$.** (Absolutely continuous and singular subspaces)

Let μ be a finite Borel measure on $\mathbb{T}$, and $E \subset L^2(\mu)$ with $zE \subset E$. We consider invariant subspaces of $L^2(\mu)$ which are based on Lebesgue decomposition of μ. A measure μ is called **absolutely continuous** with respect to m if $m(B) = 0$ implies $\nu(B) = 0$, where $B \in \mathcal{B}$ and we write $\nu \ll m$. By Radon-Nikodym theorem, there exists a positive integrable function w such that $d\nu = wdm$. That is,

$$\int_\mathbb{T} f d\nu = \int_\mathbb{T} fwdm$$

for each Borel measurable function f on $\mathbb{T}$.

A measure ν is called **singular** with respect to m if it is concentrated on a set C of Lebesgue measure zero. That is, $\nu \perp m$ if $\nu(B) = \nu(B \cap C)$ for every $B \in \mathcal{B}(\mathbb{T})$. Let μ be a finite and positive Borel measure on $\mathbb{T}$, then by Lebesgue decomposition,

$$\mu = \mu_a + \mu_s, \text{ where } \mu_a \ll m \text{ and } \mu_s \perp m.$$

So, if $f \in L^2(\mu)$, then

$$\int_\mathbb{T} |f|^2 d\mu = \int_\mathbb{T} |f|^2 d\mu_a + \int_\mathbb{T} |f|^2 d\mu_s$$

By this, we can construct an orthogonal decomposition of f. Let σ be the concentration set for μ_s. Then

$$L^2(\mu_s) \subset L^2(\mu) \text{ and } L^2(\mu_a) \subset L^2(\mu) \text{ and } L^2(\mu_s) \perp L^2(\mu_a). \tag{1.4}$$

Now, $f = f\chi_{\mathbb{T}\setminus\sigma} + f\chi_\sigma = f_a + f_s$. This means

$$L^2(\mu) = L^2(\mu_a) \oplus L^2(\mu_s). \tag{1.5}$$

The subspaces $L^2(\mu_a)$ and $L^2(\mu_s)$ are invariant subspaces and are known as absolutely continuous and singular spaces, respectively.

We need the following results in order to prove the main result about invariant subspaces of $L^2(\mu)$.

Lemma 1.9. *Let μ be a finite complex Borel measure on $\mathbb{T}$.*

(i) If $\widehat{(d\mu)}(n) = \int_{\mathbb{T}} e^{-int} d\mu(t) = 0$, for all $n \in \mathbb{Z}$, then $\mu = 0$.

(ii) If $\widehat{(d\mu)}(n) = 0$, for all $n \in \mathbb{Z} \setminus \{0\}$, then $d\mu = cdm$.

Proof. (i) Let $f \in C^2(\mathbb{T})$, then f is Borel measurable and we have

$$\begin{aligned} T_\mu(f) &= \int_{\mathbb{T}} f(t) d\mu(t) \\ &= \int_{\mathbb{T}} \Big(\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{int} \Big) d\mu(t) \\ &= \sum_{n\in\mathbb{Z}} \hat{f}(n) \int_{\mathbb{T}} e^{int} d\mu(t) \text{ (By Fubini's Theorem)} \\ &= 0 \text{ (By assumption).} \end{aligned}$$

Hence $T_\mu(f) = 0$ for all $f \in C^2(\mathbb{T})$. Since $C^2(\mathbb{T})$ is dense in $C(\mathbb{T})$, by Theorem 0.2, we get $\mu = 0$.

(ii) From the given condition and similar to the proof of case (i), we can write

$$\int_{\mathbb{T}} f(t) d\mu(t) = \hat{f}(0) \int_{\mathbb{T}} d\mu = \mu(\mathbb{T}) \int_{\mathbb{T}} f(t) dt.$$

Thus, $d\mu = \mu(\mathbb{T}) dm$, where $dm = dt$. □

Let $T : H \to H$ be an isometry (or $T \in \text{iso}(H)$) on the Hilbert space H. A subspace D of H is called **wandering** if $T^m D \perp T^n D$ for $m \neq n \, (m, n \geq 0)$.

Lemma 1.10. (H. Wold, A. Kolmogorov) *Suppose $T \in iso(H)$ and $TE \subset E$. Let $D = E \ominus TE$. Then D is a wandering subspace of T, and $E = \Big(\sum_{n\geq 0} \oplus T^n D \Big) \oplus \Big(\bigcap_{n \geq 0} T^n E \Big) = E_0 \oplus E_\infty$, where $T|_{E_\infty}$ is unitary, and $T|_{E_0}$ is completely non-unitary (i.e. if $E' \subset E_0$ and $TE' \subset E'$ implies $T|_{E'}$ is not unitary).*

Theorem 1.11. (H. Helson 1964) *Let $d\mu = wdm + d\mu_s$ be the Lebesgue decomposition of a positive finite Borel measure μ and let $E \subset L^2(\mu)$ be simply invariant. Then there exists $\sigma \subseteq \mathbb{T}$ with $m(\sigma) = 0$ and a measurable function Θ such that*

$$E = E_0 \oplus E_\infty = \Theta H^2 \oplus \chi_\sigma L^2(\mu_s),\ \text{where}$$
$$\Theta H^2 \subset L^2(\mu_a),\ \chi_\sigma L^2(\mu_s) \subset L^2(\mu_s)\ \text{and}$$
$$|\Theta|^2 w \equiv 1. \tag{1.6}$$

Conversely, if σ is measurable and Θ verified (1.6), then $\Theta H^2 \oplus \chi_\sigma L^2(\mu_s)$ is simply invariant.

Proof. Set $D = E \ominus zE = (zE)^\perp \neq \{0\}$ and let $E = \Big(\sum\limits_{n\geq 0} z^n D\Big) \oplus \Big(\bigcap\limits_{n\geq 0} z^n E\Big) = E_0 \oplus E_\infty$ be the Wold-Kolmogorov decomposition of E. Let $\Theta \in D$ with $\|\Theta\|_2 = 1$, then $\Theta \in E$ and $\Theta \perp zE$. This implies $z^n\Theta \in zE$, $\forall\, n \geq 1$, and hence $z^n\Theta \perp \Theta\, \forall\, n \geq 1$. That is,
$$\int_{\mathbb{T}} (z^n\Theta)\bar{\Theta} d\mu = \int_{\mathbb{T}} |\Theta|^2 z^n d\mu = 0,\ \forall\, n \geq 1.$$
And by conjugation
$$\int_{\mathbb{T}} |\Theta|^2 \bar{z}^n d\mu = 0,\ \forall\, n \geq 1.$$
Thus, $(\widehat{|\Theta|^2 d\mu})(n) = 0$, $\forall\, n \in \mathbb{Z} \setminus \{0\}$. By Lemma 1.9 (ii), we get $|\Theta|^2 d\mu = cdm$. But, $1 = \int\limits_{\mathbb{T}} |\Theta|^2 d\mu = c \int\limits_{\mathbb{T}} dm = c$. Thus,
$$\begin{aligned} dm &= |\Theta|^2 d\mu \\ &= |\Theta|^2 d\mu_a + |\Theta|^2 d\mu_s \\ &= |\Theta|^2 wdm + |\Theta|^2 d\mu_s. \end{aligned} \tag{1.7}$$
Implies $|\Theta|^2 = 0$ a.e. μ_s on $\mathbb{T}$ (because m has no singular part) and $dm = |\Theta|^2 wdm$ implies $|\Theta|^2 w = 1$ a.e. m. By Wold-Kolmogorov Lemma 1.10, restriction $z|_{E_\infty}$ is unitary, $zE_\infty \subseteq E_\infty = E_\infty \oplus E_0$, and $z|_{E_0}$ is non-unitary on every section of E_0, etc. Thus, we conclude that $zE_\infty = E_\infty$. By Wiener theorem, $E_\infty = \chi_\sigma L^2(\mu)$ for some $\sigma \subset \mathbb{T}$. As $\Theta \in D \subset E_0 \perp E_\infty$, implies $\Theta \perp \chi_\sigma L^2(\mu)$. In particular, this implies $\int\limits_\sigma \Theta\bar{\Theta} d\mu = \int\limits_\sigma |\Theta|^2 d\mu = 0$. Hence $\Theta|_\sigma = 0$ a.e. μ. But $\Theta \neq 0$ a.e. m implies $m(\sigma) = 0$ (since $dm = |\Theta|^2 d\mu$). Thus, in

view of (1.5) we obtain

$$E_\infty = \chi_\sigma L^2(\mu) = \chi_\sigma L^2(\mu_s) \subset L^2(\mu_s).$$

We have already shown that $D \subset L^2(\mu_a)$, because $D \subset E_0 \perp E_\infty = L^2(\mu_s)$ implies $D \subset L^2(\mu_a)$. Therefore, $E_0 = \sum\limits_{n\geq 0} \oplus z^n D \subset L^2(\mu_a)$. Also, $\overline{\text{span}}\{z^n\Theta : n \geq 0\} \subset E_0$, since $\Theta \in E_0$. We claim that $E_0 = \overline{\text{span}}\{z^n\Theta : n \geq 0\}$.

On the contrary, suppose there exists $f \in E_0 \ominus \overline{\text{span}}\{z^n\Theta : n \geq 0\}$. Then $f \perp z^n\Theta$, $\forall\, n \geq 0$. Recall that $\Theta \perp zE$. But $f \in E$, implies $z^n f \in E$ and hence $z^n f \perp \Theta$, $\forall\, n \geq 1$. Thus,

$$\int f\overline{z^n\Theta} d\mu = 0 \,\forall\, n \geq 0 \text{ and } \int z^n f\bar{\Theta} d\mu = 0, \,\forall\, n \geq 1.$$

That is, $\widehat{(f\bar{\Theta}d\mu)}(n) = 0 \,\forall\, n \in \mathbb{Z}$. By Lemma 1.9(i), it implies that $f\bar{\Theta}d\mu = 0$. Since $\bar{\Theta} \neq 0$ a.e. m and $f \in E_0 \subset L^2(\mu_a)$, it follows that $f \equiv 0$. Now, by Parseval identity, it is easy to verify that

$$\overline{\text{span}}\{z^n\Theta : n \geq 0\} = \Big\{\sum_{n\geq 0} a_n z^n \Theta : \sum_{n\geq 0} |a_n|^2 < \infty\Big\}.$$

(Notice that $\{z^n\Theta\}_{n\geq 0}$ is an orthonormal set in $L^2(\mu_a)$, since $d\mu_a = w\,dm$ and $|\Theta|^2 w \equiv 1$.) Further, it is easy to see that

$$E_0 = \Theta\Big\{\sum_{n\geq 0} a_n z^n : \sum_{n\geq 0} |a_n|^2 < \infty\Big\} = \Theta H^2.$$

Indeed, $f \mapsto \Theta f$ is an isometry from $L^2(\mathbb{T}, dm)$ onto $L^2(d\mu_a) = L^2(wdm)$. That is,

$$\int_{\mathbb{T}} |f|^2 dm = \int_{\mathbb{T}} |\Theta f|^2 d\mu_a.$$

$\square$

2. First Applications

We have seen that there is one to one correspondence between simply invariant subspace of $L^2(\mu)$ with the set of measurable unimodular functions (inner functions) due to Helson's theorem. This congruence opens many possibilities to apply Hilbert space geometry and operator theory to $L^2(\mu)$ and vice-versa. Here we discuss inner-outer decomposition of the Hardy class functions, Szegö infimum, and Riesz brother's theorem for "analytic measure". That is, for which positive measure μ on $\mathbb{T}$, the "analytic half" $\mathbb{P}_+ = \operatorname{span}\{z^n : n \geq 0\}$ is dense in $L^2(\mathbb{T}, \mu)$.

2.1. **Some consequences of Helson's theorem.** Let μ be a positive Borel measure on $\mathbb{T}$ with $d\mu = wdm + d\mu_s$. Notice that if $zE \subset E \subset L^2(\mu)$, then $E = E_a \oplus E_s$, where $zE_a \subset E_a \subset L^2(\mu_a)$, because $E = \Theta H^2 + \chi_\sigma L^2(\mu_s)$, where $\Theta H^2 \subset L^2(\mu_a)$ and $\chi_\sigma L^2(\mu_s) \subset L^2(\mu_s)$.

(a) If $\mu = \mu_s$, then $zE \subset E \subset L^2(\mu_s)$, implies $zE = E$, because, by Helson's theorem 1.11, we already have $E = \chi_\sigma L^2(\mu_s)$, which is 2-invariant.

(b) Show that for $d\mu = d\mu_a = w\,dm$, the followings are equivalent:

(i) There exists E such that $zE \subsetneq E \subset L^2(\mu_a)$.

(ii) There exists Θ such that $|\Theta|^2 w = 1$ a.e. m.

(iii) $w > 0$ almost everywhere m.

(iv) m is absolutely continuous with respect to μ_a.

(c) If $d\mu = d\mu_a = w\,dm$ and $zE \subsetneq E \subset L^2(\mu_a)$, then $E = \Theta H^2$ with $|\Theta|^2 w \equiv 1$ a.e. m.

2.2. **Reducing subspaces.** Let $f \in L^2(\mu)$ and $d\mu = wdm + d\mu_s$. We look for sufficient conditions that ensure that E_f is reducing. If there exists measurable set $e \subset \mathbb{T}$ such that $m(e) > 0$ and $f|_e = 0$. Then E_f is a reducing subspace, and there exists $\sigma \subset \mathbb{T} \smallsetminus e$ such that $E_f = \chi_\sigma L^2(\mu)$. In fact, $\sigma = \{z \in \mathbb{T} : f(z) \neq 0\}$. On the contrary, suppose $zE_f \subsetneq E_f$. Then by Theorem 1.11 we get $E_f = \Theta H^2 \oplus \chi L^2(\mu_s)$, and hence $f \in E_f$ implies $f = f_a + f_s$, where $f_a = \Theta h$, $h \neq 0$ a.e. m (by Theorem 1.8, since $h \in H^2$). This implies

$f_a \neq 0$ a.e. m, which is impossible because $f|_e = 0$ and $m(e) > 0$ implies $f_a|_e = 0$ with $m(e) > 0$. Thus, $E_f = zE_f = \chi_\sigma L^2(\mu)$ for $\sigma \subset \mathbb{T}$ (by Wiener theorem). Notice that $E_f = \overline{\text{span}}\{z^n \chi_{\mathbb{T}\setminus e} f : n \geq 0\} = \chi_{\mathbb{T}\setminus e} E_f = \chi_\sigma L^2(\mu)$ and $1 \in L^2(\mu)$, implies $\sigma \subset \mathbb{T} \setminus e$. Indeed, $\sigma = \{z \in \mathbb{T} : f(z) \neq 0\}$, which is defined up to a set of μ measure zero.

2.3. **The problem of weighted polynomial approximation.** We know that the space of trigonometric polynomials $\mathbb{P} = \text{span}\{z^n : n \in \mathbb{Z}\}$ is dense in $L^p(\mu)$ for every positive and finite measure μ and $1 \leq p < \infty$. Let $\mathbb{P}_+ = \text{span}\{z^n : n \geq 0\}$. One of the main problems is describing the closure of $\mathbb{P}_+$ in $L^2(\mu)$. Denote $H^2(\mu) = \text{clos}\, \mathbb{P}_+|_{L^2(\mu)}$. The most important part of this problem is to distinguish between the completeness case $H^2(\mu) = L^2(\mu)$, from the incompleteness case $H^2(\mu) \subsetneq L^2(\mu)$.

Corollary 2.1. $H^2(\mu) = H^2(\mu_a) \oplus L^2(\mu_s)$.

Proof. $H^2(\mu) = \overline{\text{span}}\{z^n : n \geq 0\}$. By Helson decomposition $H^2(\mu) = E_a \oplus E_s$ with $E_a \subset L^2(\mu_a)$ and $E_s \subset L^2(\mu_s)$. Since we know that $zE_s = E_s$, by Wiener theorem, $E_s = \chi_\sigma L^2(\mu_s)$ with $m(\sigma) = 0$. Since $1 \in H^2(\mu)$, we have $1 = 1_a + 1_s$ with $1_s \neq 0$ a.e. μ_s. But $1_s \in E_s = \chi_\sigma L^2(\mu_s)$ implies $\chi_\sigma L^2(\mu_s) = L^2(\mu_s)$.

Further, $(\mathbb{P}_+)_a \subset E_a$ implies $\text{clos}\,(\mathbb{P}_+)_a = H^2(\mu_a) \subseteq E_a$. But, for $f \in E_a \subset H^2(\mu)$, implies there exists $p_n \in \mathbb{P}_+$ such that $\|f - p_n\|_{L^2(\mu)} \to 0$. Since $\|f - p_n\|^2_{L^2(\mu)} = \|f - p_n\|^2_{L^2(\mu_a)} + \|p_n\|^2_{L^2(\mu_s)}$, we get $f \in H^2(\mu_a)$. (Since $f = 0$ a.e. μ_s.)

$\square$

Remark 2.2. Note that for $H^2(\mu_a)$, the closure of $\mathbb{P}_+$ in $L^2(\mu_a)$ has two possibilities:

(i) $zH^2(\mu_a) = H^2(\mu_a)$ and hence by Wiener theorem $H^2(\mu_a) = \chi_\sigma L^2(\mu_a) = L^2(\mu_a)$, because $1_a \in H^2(\mu_a)$ implies that there does not exist $\sigma \subset \mathbb{T}$ such that $m(\mathbb{T} \setminus \sigma) > 0$.

(ii) $zH^2(\mu_a) \subsetneq H^2(\mu_a) (\subset L^2(\mu_a))$, and hence $H^2(\mu_a) = \Theta H^2$ with $|\Theta|^2 w \equiv 1$.

The following results help to distinguish the above two cases.

Lemma 2.3. *$H^2(\mu)$ is reducing (and hence $H^2(\mu) = L^2(\mu)$) if and only if $\bar{z} \in H^2(\mu)$.*

Proof. If $H^2(\mu)$ is reducing, then $\bar{z} \in H^2(\mu)$ is trivial. Suppose $\bar{z} \in H^2(\mu)$, then exists $p_n \in \mathbb{P}_+$ such that $\|\bar{z} - p_n\|_{L^2(\mu)} \to 0$. Let $q \in \mathbb{P}_+$. Then

$$\int_{\mathbb{T}} |\bar{z}q - qp_n|^2 d\mu \leq \|q\|_\infty^2 \int_{\mathbb{T}} |\bar{z} - p_n|^2 \to 0 \text{ as} n \to \infty.$$

This implies $\bar{z}\,\mathbb{P}_+ \subset H^2(\mu)$, or $\mathbb{P}_+ \subset zH^2(\mu)$ (closed). Hence $H^2(\mu) \subseteq zH^2(\mu)$, i.e. $\bar{z}H^2(\mu) \subseteq H^2(\mu)$. But $zH^2(\mu) \subset H^2(\mu)$ implies $zH^2(\mu) = H^2(\mu)$. Now, it is clear from Wiener theorem and theorem 1.8 that $H^2(\mu) = \chi_\sigma L^2(\mu) = L^2(\mu)$. □

Corollary 2.4. *$H^2(\mu) = L^2(\mu)$ if and only if dist $(1, H_0^2(\mu)) = 0$, where $H_0^2(\mu)$ is the closure of span$\{z^n : n \geq 1\}$ in $L^2(\mu)$.*

Proof. Let $H^2(\mu) = L^2(\mu)$, then $\bar{z} \in H^2(\mu)$, implies $\operatorname{dist}(1, H_0^2(\mu)) = \operatorname{dist}(\bar{z}, H^2(\mu)) = 0$. On the other hand, if $\operatorname{dist}(1, H_0^2(\mu)) = 0$, then $\bar{z} \in H^2(\mu)$, and hence $H^2(\mu) = L^2(\mu)$. □

Note that the quantity

$$\operatorname{dist}^2(1, H_0^2(\mu)) = \inf_{p \in \mathbb{P}_+^0} \int_{\mathbb{T}} |1-p|^2 d\mu$$

is known **Szegö infimum**, where $\mathbb{P}_+^0 = \operatorname{span}\{z^n : n \geq 1\}$.

It can be seen that $\operatorname{dist}(1, H_0^2(\mu))$ depends only on the absolute part of the measure μ. Let $d\mu = wdm + d\mu_s$ be the lebesgue decomposition of μ. As similar to Corollary 2.1, it can be seen that $H_0^2(\mu) = H_0^2(\mu_a) \oplus L^2(\mu_s)$. We also use the fact that if M_1 and M_2 are subspaces of a Hilbert space H such that $M_1 \perp M_2$, then $P_{M_1 \oplus M_2} = P_{M_1} + P_{M_2}$ for $M_1 \perp M_2$. Thus, we can write

$$\begin{aligned}
\operatorname{dist}^2(1, H_0^2(\mu)) &= \|P_{H_0^2(\mu)} \perp 1\|_{L^2(\mu)}^2 \\
&= \|(P_{H_0^2(\mu_a)} \oplus P_{L^2(\mu_s)}) \perp (1_a + 1_s)\|_{L^2(\mu)} \\
&= \|P_{H_0^2(\mu_a)} \perp 1_a\|_{L^2(\mu_a)}^2 \,(\text{since} 1_s \in L^2(\mu_s)) \\
&= \inf_{p \in \mathbb{P}_+^0} \int_{\mathbb{T}} |1-p|^2 wdm.
\end{aligned}$$

The evaluation of Szegö infimum is intimately related to the multiplicative structure of H^2.

2.4. **The inner-outer factorization.** Recall that a function $f \in H^2$ is called **inner** if $|f| = 1$ a.e. on $\mathbb{T}$. On the other hand, $f \in H^2$ is called **outer** if $E_f = H^2$.

Theorem 2.5. (V. Smirnov, 1928) *Let $f \in H^2$ and $f \not\equiv 0$, then there exists an inner function $f_{inn} \in H^2$ and an outer function $f_{out} \in H^2$ such that $f = f_{inn} f_{out}$. Moreover, this factorization is unique and $E_f = f_{inn} H^2$.*

Proof. Note that $E_f \subset H^2$, $E_f \neq \{0\}$, and E_f is not reducing, else $\bar{z} \in H^2$. Here, $E_f = \overline{\text{span}}\{z^n f : n \geq 0\} \subset H^2$. By Theorem 1.5, we have $E_f = \Theta H^2$, where $|\Theta| = 1$ a.e. m. Let $f_{inn} = \Theta$, then $f = \Theta g$, where $g \in H^2$. We claim $E_g = H^2$. Let $h \in H^2$. Since $E_f = \Theta H^2$ and $\Theta h \in \Theta H^2$, there exists $p_n \in \mathbb{P}_+$ such that $p_n \Theta g = p_n f \to \Theta h$ inL^2. But, multiplication by an inner function is an isometry, we get

$$\|p_n g - h\|_2 = \|\Theta(p_n g - h)\|_2 \to 0.$$

Hence, $E_g = H^2$. Here $g = f_{out}$ is desired outer function.

Uniqueness: Take $f = f_1 f_2$, where f_1 is inner and f_2 is outer. As f_1 is inner, $h \mapsto f_1 h$ is an isometry, and hence as $E_{f_2} = H^2$, we get

$$f_{inn} H^2 = E_f = \overline{\text{span}}\{z^n f_1 f_2 : n \geq 0\} = f_1 \overline{\text{span}}\{z^n f_2 : n \geq 0\} = f_1 H^2.$$

By the uniqueness of the representing inner function of the simply invariant space E_f (cf. Theorem 1.5 and Corollary 1.6), we get $f_{inn} = \lambda f_1$ with $|\lambda| = 1$, and $\lambda f_1 f_{out} = f_1 f_2$ implies $f_{out} = \bar{\lambda} f_2$. $\square$

2.5. **Arithmetic of inner functions.**

Definition 2.6. Let Θ_1, Θ_2 be two inner functions in H^2. We say Θ_1 divides Θ_2 if $\frac{\Theta_2}{\Theta_1} \in H^2$.

Equivalently, Θ_1 divides Θ_2 if and only if $\Theta_1 H^2 \supset \Theta_2 H^2$. For this, if $\Theta_2 = \Theta\Theta_1$, then Θ is necessarily inner, and $\Theta_2 H^2 = \Theta_1 \Theta H^2 \subset \Theta_1 H^2$, since $\Theta H^2 \subset H^2$. On the other hand, if $\Theta_1 H^2 \supset \Theta_2 H^2$, then we get $\Theta_2 \in \Theta_1 H^2$ implies $\Theta = \frac{\Theta_2}{\Theta_1} \in H^2$.

We deduce the following two elementary properties:

Let $\Theta = gcd\{\Theta_1, \Theta_2\}$, the greatest common divisor of Θ_1 and Θ_2. Then

(i) span $\{\Theta_1 H^2, \Theta_2 H^2\} = \Theta H^2$

(ii) $\Theta_1 H^2 \cap \Theta_2 H^2 = \tilde{\Theta} H^2$, where $\tilde{\Theta} = lcm\{\Theta_1, \Theta_2\}$.

Proof. (i) $\Theta_k H^2 \subset \text{span}\{\Theta_1 H^2, \Theta_2 H^2\} = \Theta H^2\,;\, k = 1, 2$ for some inner function Θ (by Beurling's theorem) implies Θ divides $\Theta_k\,;\, k = 1, 2$. Let Θ' be another divisor of $\Theta_k : k = 1, 2$. Then $\Theta' H^2 \supset \Theta_k H^2$, and hence $\Theta' H^2 \supset \text{span}\{\Theta_k H^2;\, k = 1, 2\} = \Theta H^2$. This implies Θ' divides Θ and thus $\Theta = gcd\{\Theta_k;\, k = 1, 2\}$. The proof of (ii) is similar to (i). □

Definition 2.7. Let $\{\Theta_i : i \in I\}$ be a family of inner functions.

(i) $\Theta = \text{gcd}\{\Theta_i : i \in I\}$ if Θ divides each Θ_i, and Θ is divisible by every other inner function that divides each Θ_i.

(ii) $\Theta = lcm\{\Theta_i : i \in I\}$ if each Θ_i divides Θ and Θ divides every other inner function that is divisible by each Θ_i

Convention: In case the gcd or the lcm does not exist, we write $gcd\{\Theta_i : i \in I\} = 1$ and $lcm\{\Theta_i : i \in I\} = 0$.

Corollary 2.8. *span*$\{\Theta_i \in H^2 : i \in I\} = \Theta H^2$, *where* $\Theta = gcd\{\Theta_i : i \in I\}$ *and* $\cap \Theta_i H^2 = \tilde{\Theta} H^2$, *where* $\tilde{\Theta} = lcm\{\Theta_i : i \in I\}$.

Corollary 2.9. *Let F be a proper subset of H^2. Then* $\overline{span}\{z^n F : n \geq 0\} = \Theta H^2$, *where* $\Theta = gcd\{f_{inn} : f \in F \setminus \{0\}\}$, *and f_{inn} stands for inner factor of f.*

Proof. We have $\overline{\text{span}}\{z^n F : n \geq 0\} = \overline{\text{span}}\{f_{inn} H^2 : f \in F \setminus \{0\}\}$. (By Smirnov's theorem). By applying Corollary 2.8 we get the required. □

2.6. **Characterization of outer functions.**

Theorem 2.10. *Let $f \in H^2$. Then the followings are equivalent:*

(i) f is outer

(ii) f is a divisor of the space H^2, i.e. if $g \in H^2$ and $\frac{g}{f} \in L^2$, then $\frac{g}{f} \in H^2$.

Proof. (ii) $\Longrightarrow$ (i): Let $f = f_{inn} f_{out}$ be an inner-outer factorization of f. Then $\bar{f}_{inn} = \frac{1}{f_{inn}} = \frac{f_{out}}{f} \in L^2$ because of $f_{inn} \in H^2 \subset L^2$. By (ii), we get $\bar{f}_{inn} \in H^2$. But $f_{inn} \in H^2$ implies $\bar{f}_{inn} = \lambda$ (constant) with $|\lambda| = 1$. Hence $f = \bar{\lambda} f_{out}$.

(i) $\Longrightarrow$ (ii): Given f is outer, we have $E_f = H^2$. Since $1 \in H^2$, there exists $p_n \in \mathbb{P}_+$ such that $p_n f \to 1$ in L^2. Let $g \in H^2$ and $h = \frac{g}{f} \in L^2$. Then

$$\int_{\mathbb{T}} |p_n g - h| = \int_{\mathbb{T}} |p_n f - 1||h| \leq \|p_n f - 1\|_2 \|h\|_2 \to 0 \text{ as} n \to \infty. \tag{2.1}$$

But $p_n g \in H^2$, implies $\widehat{(p_n g)}(k) = 0$ if $k < 0$. Since $\varphi \mapsto \hat{\varphi}(k)$ is continuous linear functional on $L^1(\mathbb{T})$ for each k, by (2.1) we get $(\hat{h})(k) = 0$, $\forall\, k < 0$. Thus, $h \in H^2$. $\square$

Corollary 2.11. *If two outer functions f_1 and f_2 verify $|f_1| = |f_2|$ a.e. on $\mathbb{T}$, then $f_1 = \lambda f_2$ where $|\lambda| = 1$.*

Proof. Since f_2 is outer, $f_1 \in H^2$, and $|\frac{f_1}{f_2}| = 1 \in L^2$, by Theorem 2.10 (ii), we get $\frac{f_1}{f_2} \in H^2$. In the similar way $\overline{\frac{f_1}{f_2}} = \frac{f_2}{f_1} \in H^2$ implies $\frac{f_1}{f_2} = \lambda$ (constant) and hence $f_1 = \lambda f_2$ with $|\lambda| = 1$. Thus, an outer function is completely defined by its modulus. $\square$

Corollary 2.12. *Let $w \geq 0$, $w \in L^1(\mathbb{T})$. If there exists $f \in H^2$ such that $|f|^2 = w$ a.e. $\mathbb{T}$, then there exists a unique outer function $f_0 \in H^2$ such that $|f_0|^2 = w$ a.e. $\mathbb{T}$.*

(Hint: By Smirnov theorem, $f = f_{inn} f_{out}$ etc.)

2.7. **Szegö infimum and Riesz Brother's theorem.** Here we consider two theorems in two different settings by using the fact that in an orthogonal complement of the analytic polynomials $\mathbb{P}_+$ the absolute component of a measure is only important.

Theorem 2.13. (Szegö and Kolmogorov) *Let μ be a finite Borel measure on $\mathbb{T}$ with Lebesgue decomposition $d\mu = w dm + d\mu_s$, where $w \in L^1_+(\mathbb{T})$.*

(i) If there does not exist $f \in H^2$ such that $|f|^2 = w$ a.e. m, then

$$\inf_{p \in \mathbb{P}^0_+} \int_{\mathbb{T}} |1 - p|^2 d\mu = 0.$$

(ii) If there exists $f \in H^2$ such that $|f|^2 = w$ a.e. m, and f is outer, then

$$\inf_{p \in \mathbb{P}_+^0} \int_{\mathbb{T}} |1-p|^2 d\mu = |\hat{f}(0)|^2.$$

Proof. (ii). We know that the Szegö infimum I will satisfy

$$\begin{aligned} I^2 = \operatorname{dist}^2(1, H_0^2(\mu)) &= \operatorname{dist}^2(1, H_0^2(\mu_a)) \\ &= \inf_{p \in \mathbb{P}_+^0} \int_{\mathbb{T}} |1-p|^2 w dm. \end{aligned}$$

Given that $|f|^2 = w$ a.e. m, and f is outer. Hence

$$I^2 = \inf_{p \in \mathbb{P}_+^0} \int_{\mathbb{T}} |f - pf|^2 dm.$$

As f is an outer function, we can verify that $\overline{\operatorname{span}}\{z^n f : n \geq 1\} = zH^2$. Hence $I = \operatorname{dist}_{H^2}(f, zH^2)$. Note that $f = \sum_{n \geq 0} \hat{f}(n) z^n = \hat{f}(0) + g$, where $g \in zH^2$. Since $\hat{f}(0) \perp zH^2$, it follows that $I = \operatorname{dist}_{H^2}(\hat{f}(0), zH^2) = |\hat{f}(0)|$.

(i). Now, we consider the invariant space $E_a = H_0^2(\mu_a)$. If $zE_a \neq E_a$, then there exists Θ such that $E_a = \Theta H^2$ with $|\Theta|^2 w \equiv 1$. But $z \in E_a$ and hence $z = \Theta f$ for some $f \in H^2$. This implies that $|f|^2 = \frac{1}{|\Theta|^2} = w$ (since $|z| = 1$), and this leads to case (ii). Hence, case (i) is possible only if $zE_a = E_a$. But, then $E_a = L^2(\mu_a)$ by Remark 2.2(i). Hence $\operatorname{dist}(1, H_0^2(\mu)) = 0$, since $1 \in L^2(\mu_a) = H_0^2(\mu_a)$. □

The above Theorem (Szegö and Kolmogorov) leads to the problem of computing $|\hat{f}(0)|^2$ in terms of w. In order to do this, we have to consider H^2 as a space of analytic functions on the unit disc, which we do later.

Riesz Brother's result is an important consequence of Helson's theorem. For that, we need to recall an important result related to the Radon-Nikodym derivative.

Let $|\mu|$ be the total variation measure of a complex-valued Borel measure μ on $\mathbb{T}$, i.e.

$$|\mu|(\sigma) = \sup \Big\{ \sum_{i \in I} |\mu(\sigma_i)| : \{\sigma_i\}_{i \in I} \text{ is a partition of} \sigma \text{ in} \mathcal{B}(\mathbb{T}) \Big\}.$$

Suppose μ is absolutely continuous with respect to a positive measure λ on $\mathcal{B}(\mathbb{T})$. Then there exists $\varphi \in L^1(\lambda)$ (the Radon-Nikodym derivative of μ with respect to λ) such that

$$|\mu|(\sigma) = \int_{\sigma} |\varphi| d\lambda.$$

Theorem 2.14. (Riesz Brother's, 1916) *Let μ be a complex-valued Borel measure on $\mathbb{T}$ such that*

$$\int_{\mathbb{T}} z^n d\mu = 0, \, \forall\, n \geq 1.$$

Then $\mu \ll m$ and $d\mu = h\, dm$, where $h \in H^1 = \{f \in L^1(\mathbb{T}) : \hat{f}(k) = 0,\, k < 0\}$.

Note that, a measure μ that satisfies $\int\limits_{\mathbb{T}} \bar{z}^n d\mu = 0$ for $n < 0$ will be called analytic.

Proof. It is clear that $\mu \ll |\mu|$. Let $g \in L^1(|\mu|)$ be the corresponding Radon-Nikodym derivative of μ with respect to $|\mu|$. We claim that $|g| = 1$ a.e. μ. For $\delta > 0$, set $\sigma = \{t : |g(t)| < 1 - \delta\}$. Then $|\mu|(\sigma) = \int\limits_{\sigma} |g| d|\mu| \leq (1-\delta)|\mu|(\sigma)$. Implies $|\mu|(\sigma) = 0$. Similarly, the case $\sigma' = \{t : |g(t)| > 1 - \delta\}$. This proves the claim. As a consequence of the Corollary 2.1, we get

$$H_0^2(|\mu|) = H^2(|\mu|_a) \oplus L^2(|\mu|_s). \tag{2.2}$$

But $|g| = 1$ a.e. $|\mu|$ implies $\bar{g} \in L^2(|\mu|)$, and

$$\langle z^n, \bar{g} \rangle_{L^2(|\mu|)} = \int_{\mathbb{T}} z^n g d|\mu| = \int_{\mathbb{T}} z^n d\mu = 0, \, n \geq 1.$$

In other words, $\bar{g} \perp z^n$, $n \geq 1$ in the Hilbert space $L^2(|\mu|)$, and hence $\bar{g} \perp H_0^2(|\mu|)$. In view of (2.2), we obtain $\bar{g} \perp H_0^2(|\mu|_s)$. Now, by construction, $|g| = 1$ a.e. $|\mu|$, which implies $|g| = 1$ a.e. $|\mu|_s$. This is impossible (since $\bar{g} \perp H_0^2(|\mu|_s)$), unless $|\mu|_s = 0$. Finally, $\mu \ll |\mu|$ implies

$$\mu(\sigma) = \int_{\sigma} g d|\mu| = \int_{\sigma} g d|\mu|_a = \int_{\sigma} g w dm$$

for each $\sigma \in \mathcal{B}(\mathbb{T})$. That is, $\mu \ll m$ with Radon-Nikodym derivative $h = gw \in L^1(\mathbb{T})$, and

$$\hat{h}(k) = \int_{\mathbb{T}} \bar{z}^k h dm = \int_{\mathbb{T}} \bar{z}^k g d|d\mu| = \int_{\mathbb{T}} \bar{z}^k d\mu = 0 \ \text{ if} k \leq -1.$$

Hence $h \in H^1$. □

Question 2.15. *

For $g \in L^1(\mathbb{T})$, define $g_f = \overline{\text{span}}\{z^n g : n \geq 0\}|_{L^1(\mathbb{T})}$. Characterize all possible $g \in L^1(\mathbb{T})$ such that $\inf\limits_{p \in P_+^0} \|1 - p\, g\|_1 = 0$.

3. Canonical factorization of H^p-spaces on disc

In this section, we discuss the canonical factorization of functions in H^p- spaces on the open unit disc as a product of three factors, namely a Blaschke product, a singular inner function, and an outer function in its Schwarz-Herglotz representation. This will help us analyze the questions raised earlier. In particular, Szegö infimum etc.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathrm{Hol}(\mathbb{D})$ denotes the space of analytic functions on $\mathbb{D}$. For $p > 0$, set

$$H^p(\mathbb{D}) = \left\{ f \in \ \mathrm{Hol}(\mathbb{D}) : \|f\|_{H^p}^p = \sup_{0\le r<1} \int_0^{2\pi} |f(re^{it})|^p dt < \infty \right\},$$

and $H^\infty(\mathbb{D}) = \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)| < \infty\}$. Here dt is the normalized measure on $\mathbb{T}$.

For $p \geq 1$, set $L^p = L^p[0, 2\pi] = (L^p[0, 2\pi], dt)$ and $H^p = \{f \in L^p : \hat{f}(k) = 0;\ k < 0\}$.

The space $H^p(\mathbb{D})$ and H^p are called **Hardy spaces of the disc** and **Hardy space** respectively. Later on we canonically identify these two spaces as same.

3.1. **Straight forward properties:**

(i) $H^p(\mathbb{D})$ is a linear space.

(ii) $f \longmapsto \|f\|_{H^p}$ is a norm if $p \geq 1$.

(iii) $H^p(\mathbb{D}) \subset H^q(\mathbb{D})$ if $p > q$.

(iv) For $p = 2$, let $f \in \mathrm{Hol}(\mathbb{D})$, and

$$f(z) = \sum_{n\ge0} \hat{f}(n) z^n,\ \hat{f}(n) \in \mathbb{C}.$$

By Parseval's identity

$$\int_0^{2\pi} |f(re^{it})|^2 dt = \sum_{n\ge0} |\hat{f}(n)|^2 r^{2n},\ 0 \le r < 1$$

and we have

$$\sup_{0\le r<1} \int_0^{2\pi} |f(re^{it})|^2 dt = \sum_{n\ge0} |\hat{f}(n)|^2.$$

Thus, for $f \in \mathrm{Hol}(\mathbb{D})$, we have $f \in H^2(\mathbb{D})$ if and only if $\sum_{n\ge0} |\hat{f}(n)|^2 < \infty$.

3.2. **A revisit to Fourier series:** The functions in $L^p[0,2\pi]$ can be thought of as functions on $(0,2\pi)$, which can be extended periodically to real line $\mathbb{R}$.

Lemma 3.1. *Let $f \in L^1[0,2\pi]$, $g \in L^p[0,2\pi]$, $1 \leq p \leq \infty$. Then*

(i) for almost every $x \in (0,2\pi)$, $y \longmapsto f(x-y)g(y)$ is integrable on $(0,2\pi)$.

(ii) $f * g(x) = \int_0^{2\pi} f(x-y)g(y)dy$ is well defined and belongs to $L^p[0,2\pi]$.

(iii) $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Proof. Note that $(x,y) \longmapsto f(x-y)g(y)$ is measurable, and by Fubini's theorem $|f * g(x)| \leq \int |f(x-y)||g(y)|dy < \infty$ a.e. x. By Minkowski integral inequality,

$$\Big|\Big| \int f(x-y)g(y)dy \Big|\Big|_p \leq \int \|f(x-y)g(y)\|_p dy = \|g\|_p \|f_1\|.$$

Further, if $f \in L^1(0,2\pi)$ and $\hat{f}(n) = \int_0^{2\pi} f(t)e^{-int}dt$, then $\widehat{(f * g)}(n) = \hat{f}(n)\hat{g}(n)$, whenever $g \in L^p$ and $1 \leq p \leq \infty$. (Using Fubini's theorem) □

3.3. **Approximation identity (or good kernel).**

(i) If a family $(E_\alpha) \subset L^1$ satisfies

(a) $\sup\limits_\alpha \|E_\alpha\|_1 < \infty$

(b) $\lim\limits_\alpha \hat{E}_\alpha(n) = 1$,

then $\lim\limits_\alpha \|f - f * E_\alpha\|_p = 0$ for $f \in L^p (1 \leq p < \infty)$. This is still true for $p = \infty$, if $f \in C(\mathbb{T})$.

(ii) If $(E_\alpha) \subset L^1$ satisfies

(a) $\sup\limits_\alpha \|E_\alpha\|_1 < \infty$

(b) $\lim\limits_\alpha \displaystyle\int_0^{2\pi} E_\alpha dx = 1$

(c) $\lim\limits_\alpha \sup\limits_{\delta < |x| < \pi} |E_\alpha(x)| = 0 \, \forall \, \delta > 0$.

then conditions of (a) and (b) of (i) is satisfied and we get $\lim\limits_\alpha \|f - f * E_\alpha\|_p = 0$.

3.4. **Dirichlet, Fejer and Poisson Kernels:** (i) Dirichlet kernel

$$D_m = \sum_{k=-m}^{m} e^{ikt} = \frac{\sin(m+\frac{1}{2})t}{\sin(t/2)}.$$

(ii) Fejer kernel

$$\Phi_n(t) = \frac{1}{n+1}\sum_{m=0}^{n} D_m = \sum_{k=-n}^{n}\Big(1-\frac{|k|}{n+1}\Big)e^{ikt} = \frac{1}{n+1}\Big(\frac{\sin\frac{n+1}{2}t}{\sin(t/2)}\Big)^2.$$

(iii) Poisson kernel

$$P_r(t) = P(re^{it}) = \frac{1-r^2}{|1-re^{it}|^2} = \sum_{k\in\mathbb{Z}} r^{|k|}e^{ikt},\ 0\le r<1.$$

Result: If $f \in L^1$, then

(1) $f * D_m(t) = \displaystyle\sum_{k=-m}^{m}\hat{f}(k)e^{ikt} = S_m(f;t)$ (Partial Fourier series sums of f)

(2) $f * \Phi_n(t) = \displaystyle\sum \hat{f}(j)\Big(1-\frac{|j|}{n+1}\Big)e^{ijt} = \frac{1}{n+1}\sum_{m=0}^{n} S_m(f;t)$ (Arithmetic mean of partial sum of Fourier series of f)

(3) $f * P_r(t) = \displaystyle\sum_{k\in\mathbb{Z}}\hat{f}(k)r^{|k|}e^{ikt}$, $0\le r<1$.

(4) $(\Phi_n)_{n\ge1}$ and $(P_r)_{0\le r<1}$ are good kernels, and $\|P_r\|_1 = \|\Phi_n\|_1 = 1$.

(5) $P_r * P_{r'} = P_{rr'}$ for $0\le r, r'<1$ (semi group property).

Corollary 3.2. *If $f\in L^p$, $1\le p<\infty$, then $\lim_{n\to\infty}\|f - f*\Phi_n\|_p = 0$. Hence trigonometric polynomials are dense in L^p. (Hint: This follows from the property of the good kernel.)*

The same is true for $p=\infty$, if $f\in C(\mathbb{T})$.

Corollary 3.3. *If $f\in L^1$, $\hat{f}(n)=0$, $\forall\, n\in\mathbb{Z}$, then $f=0$.*

Notations: For $f\in L^1$, set $f_r = f*P_r$, $0\le r<1$. For $f\in \mathrm{Hol}(\mathbb{D})$, we set $f_{(r)}(z) = f(rz)$, if $|z|<\frac{1}{r}$, $0\le r<1$.

Corollary 3.4. *If $0\le r<\rho<1$ and $f\in L^p$, $1\le p<\infty$, then $\lim_{r\to1}\|f-f_r\|_p = 0$. Moreover, $\|f_r\|_p\le\|f_\rho\|_p\le\|f\|_p$.(Using maximum modulus principle.) If $f\in Hol(\mathbb{D})$, then $\|f_{(r)}\|_p\le\|f_{(\rho)}\|_p$ and $\lim_{r\to1}\|f_{(r)}\|_p\le\infty$. In fact, $\lim_{r\to1}\|f_{(r)}\|_p = \|f\|_{H^p(\mathbb{D})}$ if $f\in H^p(\mathbb{D})$. (It follows due to P_r is a good kernel.)*

3.5. **Identification of $H^p(\mathbb{D})$ with $H^p(\mathbb{T})$.**

Theorem 3.5. *Let $1\le p\le\infty$,* *(i) If $f \in H^p(\mathbb{D})$, then $\lim\limits_{r\to 1} f_{(r)} = \tilde{f}$ exists in $L^p(\mathbb{T})$ and $\tilde{f} \in H^p(\mathbb{T})$. (For $p = \infty$, the limit holds in the weak* topology of $L^\infty(\mathbb{T})$ i.e. in $\sigma(L^\infty, L^1)$.)*

(ii) $f \longmapsto \tilde{f}$ is an isometry.

(iii) f and $\tilde{f}$ are related by $f_{(r)} = (\tilde{f})_r = \tilde{f} * P_r$.

*Here the function $\tilde{f}$ is called the **boundary limit** of function f.*

Proof. Let $f = \sum\limits_{n=0}^{\infty} a_n z^n \in H^p(\mathbb{D})$, then

$$M = \sup_{0\le r<1} \|f_{(r)}\|_p < \infty. \tag{3.1}$$

(i) For $1 < p < \infty$, by Banach Alaoglu theorem (3.1) implies that $(f_{(r)})_{0\le r<1}$ is weakly relatively compact in $L^p(\mathbb{T})$. Since $L^p = (L^{p'})^*$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $f_{(r)} \in L^p$; $M = \sup\limits_{0\le r<1} \|\Lambda f_{(r)}\| < \infty$, where $\Lambda f_{(r)} \in (L^{p'})^*$. This gives a limit point $\tilde{f} \in L^p(\mathbb{T})$ of $(f_{(r_k)})_{r_k \to 1}$ in the weak topology of L^p. We claim that the convergence takes place in L^p. As the functional $\phi \longmapsto \hat{\phi}(n)$ is continuous on L^p, for $\epsilon > 0$, $0 < r < 1$, $\exists\, r_k$ with $r < r_k < 1$ such that $|\hat{f}_{(r)}(n) - \hat{\tilde{f}}(n)| < \epsilon$. Note that

$$\|f_{(r)} - \tilde{f}\|_p \le \|f_{(r)} - f_{(r_k)}\|_p + \|f_{(r_k)} - \tilde{f}\|_p \to 0 \text{ as} r \to 1.$$

if we suppose $f_{(r_k)} \to \tilde{f}$ in L^p. But then $\hat{f}_{(r)}(n) = a_n r^n \to a_n$, $n \in \mathbb{Z}$ with $a_n = 0$ if $n < 0$. Hence $a_n = \widehat{(\tilde{f})}(n)$ implies $f \in H^p(\mathbb{T})$.

We deduce that $\tilde{f}$ does not depends on $(r_k)_{k\ge 1}$ and for $\xi \in \mathbb{T}$,

$$(\tilde{f} * P_r)(\xi) = \sum a_n r^k \xi^n = \sum \widehat{(\tilde{f})}(n) r^{|n|} \xi^n = f_{(r)}(\xi). \tag{3.2}$$

Now, by property of good kernel P_r we get

$$\|f_{(r)} - \tilde{f}\|_p = \|(\tilde{f})_r - \tilde{f}\|_p \to 0 \text{ as} r \to 1.$$

That is $f_{(r)} \to \tilde{f}$ in L^p.

For $p = \infty$, the similar reasoning gives the convergence $f_{(r)} = (\tilde{f})_r \to \tilde{f}$ in weak* topology of L^∞.

Case $p = 1$: The space $L^1(\mathbb{T})$ can be regarded as a subspace of $\mathcal{M}(\mathbb{T})$, the space of all complex measures on $\mathbb{T}$. As $\mathcal{M}(\mathbb{T}) = C(\mathbb{T})^*$, by Banach Alaoglu theorem, the balls of $\mathcal{M}(\mathbb{T})$ are weak* relatively compact.

We again get the existence of limit $\tilde{f} \in \mathcal{M}(\mathbb{T})$ as $\lim_{r\to 1} f_{(r)} = \tilde{f}$, but this is weak* limit in $\mathcal{M}(\mathbb{T})$. That is, $\int f_{(r)} g \to \int \tilde{f} g$, $g \in C(\mathbb{T})$. As before take $g(t) = e^{-int}$, then $\widehat{(\tilde{f})}(n) = \hat{\mu}(n) = \lim_{r\to 1} \hat{f}_{(r)}(n)$, $n \in \mathbb{Z}$, and hence $\hat{\mu}(n) = 0$ if $n < 0$. By Riesz Brother's theorem we get $\mu << m$, and the corresponding Radon Nikodym derivative of μ with respect to m is equal to $\tilde{f} \in H^1$. Using the same argument as in the beginning of the proof, we get $\widehat{(\tilde{f})}(n) = a_n$, $n \geq 0$, $f_r = (\tilde{f})_r$. Hence

$$\lim_{r\to 1} \|\tilde{f} - f_{(r)}\|_1 = \|\tilde{f} - (\tilde{f})_r\|_1 \to 0$$

because $f_r \to f$ in L^p for $1 \leq p < \infty$.

(ii) Let us first consider the case $p < \infty$. Since $\tilde{f} = \lim_{r\to 1} f_{(r)}$, we get

$$\|\tilde{f}\|_p = \lim_{r\to 1} \|f_{(r)}\|_p = \|f\|_{H^p(\mathbb{D})}.$$

For $p = \infty$, observe that as $\tilde{f}$ is weak* limit of $f_{(r)}$, we get

$$\|\tilde{f}\|_\infty \leq \liminf_{r\to 1} \|f_{(r)}\|_\infty = \|f\|_{H^\infty(\mathbb{D})}.$$

On the other hand $f_{(r)} = \tilde{f} * P_r$, we get

$$\limsup_{r\to 1} \|f_{(r)}\|_\infty \leq \|\tilde{f}\|_\infty.$$

Hence, we conclude that $\|f\|_{H^\infty(\mathbb{D})} = \|\tilde{f}\|_{H^\infty(\mathbb{T})} = \|\tilde{f}\|_\infty$.

(iii) has been given in (3.1).

□

Convention: Thus, in view of Theorem 3.5, we can identify $f \in H^p(\mathbb{D})$ and its boundary limit $\tilde{f}$ by

$$f_{(r)} = f_r = f * P_r \text{ and } f = \sum_{n\geq 0} \hat{f}(n) z^n.$$

Now, $\hat{f}(n)$ represents Fourier coefficient of $\tilde{f}$ at n and Taylor's coefficient as well.

Corollary 3.6. *For $\xi \in \mathbb{D}$, $f \longmapsto f(\xi)$ is a continuous linear functional on H^1 (and hence on H^p, $1 \leq p < \infty$).*

Proof. Let $\tilde{f}$ be the boundary limit of $f \in H^1(\mathbb{D})$. Write $\xi = re^{it}$, $0 \leq r < 1$. Then

$$\tilde{f} * P_r(e^{it}) = \sum \hat{\tilde{f}}(n) e^{int} r^{|n|} = \sum a_n e^{int} r^n = f_{(r)}(e^{int}) = f(re^{int}) = f(\xi).$$

Thus $|f(\xi)| \leq \|\tilde{f}\|_1 \|P_r\|_\infty \leq \|\tilde{f}\|_1 \dfrac{1+|\xi|}{1-|\xi|}$. □

Remark 3.7. If $\tilde{f}_n \to \tilde{f}$ in $H^p, 1 \leq p < \infty$, then $f_n \to f$ uniformity on compact sets in $\mathbb{D}$.

3.6. **Jensen's formula and Jensen's inequality:**

Lemma 3.8. *Let $f \in H^1$ with $\hat{f}(0) \neq 0$(because $f(0) = \hat{f}(0)$) and let λ_n be the sequence of zeroes of f in $\mathbb{D}$ counted with multiplicity. Then*

$$\log|f(0)| + \sum_{n\geq 1} \log \frac{1}{|\lambda_n|} \leq \int_{\mathbb{T}} \log|f(t)| dm(t).$$

If $f \in Hol(\mathbb{D}_{1+\epsilon})$, then

$$\log|f(0)| + \sum_{n\geq 1} \log \frac{1}{|\lambda_n|} = \int_{\mathbb{T}} \log|f(t)| dm(t).$$

Proof. First we consider $f \in \mathrm{Hol}(\mathbb{D}_{1+\epsilon})$. Let us assume that $Z(f) \cap \mathbb{T} = \emptyset$, i.e. f has no zeroes on $\mathbb{T}$. Then $Z(f) \cap \mathbb{D}$=finite=$\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Set $B(z) = \prod_{j=1}^{n} \frac{|\lambda_j|}{\lambda_j} \frac{(\lambda_j - z)}{(1-\bar{\lambda}_j z)}$. For $B_\lambda(z) = \frac{|\lambda|}{\lambda} \frac{(\lambda - z)}{(1-\bar{\lambda} z)}$, it is easy to see that

$$|B_\lambda(z)|^2 = 1 - \frac{(1-|\lambda|^2)(1-|z|^2)}{|1-\bar{\lambda}z|^2}.$$

Thus we set $|B| = 1$ on $\mathbb{T}$, and f/B is a zero free holomorphic function on $\mathbb{D}_{1+\delta}$ for some $\delta > 0$. Hence, $\log|f/B|$ is a harmonic function on $\mathbb{D}_{1+\delta}$ and allow to apply MVT (because $\log g(z) = \log|g(z)| + i \arg(g(z))$, if $g(z) \neq 0$) and we get

$$\log|(f/B)(0)| = \int_{\mathbb{T}} \log|f/B| dm = \int_{\mathbb{T}} \log|f| dm.$$

As $\log|(f/B)(0)| = \log|(f)(0)| + \sum_{j=1}^{\infty} \log|\lambda_j|^{-1}$, we get the desired formula.

For f having zero on $\mathbb{T}$, we consider f_r, $0 \leq r < 1$. Choose r such that f_r has no zero on $\mathbb{T}$. In view of the previous case, we get

$$\log|f(0)| + \sum_{|\lambda_n| \leq r} \log \frac{r}{|\lambda_n|} = \int_{\mathbb{T}} \log|f_r| dm(t) \tag{3.3}$$

Now, f is analytic in $\mathbb{D}_{1+\epsilon}$, so f has finite number of zeros on $\mathbb{T}$. Let $Z(f) \cap \mathbb{T} = \{\xi_j : j = 1, 2, \dots, k\}$. Then in the neighborhood of each ξ_j, we have

$$\log|f(r\xi)| \leq C \log|\xi - \xi_j|^{-1}, \forall\, r;\ 0 \leq r < 1,$$

where C depends upon multiplicities of zeroes of ξ_j. (Hint: $f(r\xi) = (r\xi - \xi_j)^n g(r\xi)$, and easy to see that $\frac{1}{2}|\xi - \xi_j| \leq |r\xi - \xi_j|$ etc.) As $\log|\xi - \xi_j|^{-1}$ is integrable, we may pass to limit in (3.3).

The general case: Let $f \in H^1$ and $f(0) \neq 0$. In order to pass limit in (3.3), note that $|\log x - \log y| \leq C_\epsilon |x - y|$, if $x, y > \epsilon$. Hence

$$|\log(|f_r| + \epsilon) - \log(|f| + \epsilon)| \leq C_\epsilon ||f_r| - |f|| \text{ on} \mathbb{T} \text{ and}$$

$$\log(|f_r| + \epsilon) \to \log(|f| + \epsilon) \text{ in} L^1(\mathbb{T}) \text{ as} r \to 1.$$

But from (3.3)

$$\log|f(0)| + \sum_{|\lambda_n| \leq r} \log \frac{r}{|\lambda_n|} \leq \int_{\mathbb{T}} \log(|f_r| + \epsilon) dm(t). \tag{3.4}$$

As LHS in (3.4) is increasing in r and RHS is convergent, we obtain

$$\log|f(0)| + \sum_{n \geq 1} \log \frac{1}{|\lambda_n|} \leq \int_{\mathbb{T}} \log(|f| + \epsilon) dm$$

for each $\epsilon > 0$. This completes the proof. □

Remark 3.9. (Generalized Jensen's inequality)

Let $g \in H^1$, $g \not\equiv 0$, and $|\xi| < 1$. Then

$$\log|g(\xi)| \leq \int \frac{1 - |\xi|^2}{|\xi - t|^2} \log|g(t)| dm(t). \tag{3.5}$$

Indeed, to begin with, we may assume that $g \in$ Hol$(\mathbb{D}_{1+\epsilon})$. Apply the previous result to the function

$$f(z) = g\Big(\frac{\xi - z}{1 - \bar{\xi} z}\Big),$$

and remark that Jacobian of this change of variable is $\frac{1-|\xi|^2}{|\xi - z|}$. (Hint: Put $s = \frac{\xi - t}{1 - \bar{\xi} t}$ etc.)

3.7. **The boundary uniqueness theorem:**

Corollary 3.10. *If $g \in H^1, g \not\equiv 0$, then $\log|g| \in L^1(\mathbb{T})$. In particular, if $g \in H^1$ and $m\{t \in \mathbb{T} : g(t) = 0\} > 0$, then $g \equiv 0$.*

Proof. Indeed, $g \in H^1$ may be expanded in its Taylor's series (when realized on disc $\mathbb{D}$) as $g = \sum_{k \geq n} \hat{g}(k) z^k$, where $\hat{g}(n) \neq 0$, and $n \geq 0$ is the multiplicity of the zero at $z = 0$.

By applying Jensen's inequality to function $f = g/z^n$, we get

$$\int_{\mathbb{T}} \log |g| dm = \int_{\mathbb{T}} \log |f| dm > -\infty.$$

Since, $\log x < x$ if $x > 0$, we also have

$$\int_{\mathbb{T}} \log |g| dm \leq \int_{\mathbb{T}} |g| dm < \infty.$$

Hence, $\log |g| \in L^1(\mathbb{T})$. It is clear that if $m\{t \in \mathbb{T} : g(t) = 0\} > 0$, then $\int_{\mathbb{T}} \log |g| dm = -\infty$, which is possible only if $g \equiv 0$. □

Remark 3.11. Recall that we have seen the second statement of the above corollary for $f \in H^2$ using a completely different approach.

3.8. **Blaschke Product.**

Lemma 3.12. (*Blaschke condition, interior uniqueness theorem*) *Suppose* $f \in Hol(\mathbb{D})$, $f \not\equiv 0$, *and let* $(\lambda_n)_{n \geq 1}$ *be the zero sequence of* f *in* $\mathbb{D}$, *where each zero is repeated according to its multiplicity. Suppose that*

$$\liminf_{r \to 1} \int_{\mathbb{T}} \log |f_r| dm < \infty,$$

then $\sum_{n \geq 1} (1 - |\lambda_n|) < \infty$. *In particular, this holds whenever* $f \in H^p(\mathbb{D})$, $p > 0$.

Remark 3.13. The condition $\displaystyle\sum_{n \geq 1} (1 - |\lambda_n|) < \infty$ is called Blaschke condition.

Proof. Without loss of generality, we can assume that $f(0) \neq 0$. But then Jensen's formula gives

$$\sum_{n \geq 1} \log \frac{1}{|\lambda_n|} = \liminf_{r \to 1} \sum_{|\lambda_n| \leq r} \log \frac{r}{|\lambda_n|} < \infty$$

As $|\lambda_n| \to 1$, we have $\log \left(\frac{1}{|\lambda_n|} \right) \sim (1 - |\lambda_n|)$, and hence the desired conclusion followed. The $H^p(\mathbb{D})$ case is a consequence of the obvious estimate $\log x < C_p x^p$ for $x > 0$, $p > 0$, because

$$\liminf_{r \to 1} \int_{\mathbb{T}} \log |f_r| \leq \liminf_{r \to 1} \int_{\mathbb{T}} C_p |f_r|^p < \infty.$$

□

For $\lambda \in \mathbb{D}$, we define Blaschke factor by

$$b_\lambda(z) = \frac{|\lambda|}{\lambda} \frac{(\lambda - z)}{(1 - \bar{\lambda} z)}.$$

(i) If we assume the normalization $b_\lambda\big(-\frac{\lambda}{|\lambda|}\big) = 1$, then for $\lambda = 0$, we can define $b_0(z) = z$.

(ii) Zero set $Z(b_\lambda) = \{\lambda\}$, $b_\lambda \in$Hol$(\mathbb{C} \setminus \{\frac{1}{\bar{\lambda}}\})$, $|b_\lambda| \leq 1$ on $\mathbb{D}$ and $|b_\lambda| = 1$ on $\mathbb{T}$.

Lemma 3.14. *If $(\lambda_n)_{n\geq 1} \in \mathbb{D}$ satisfies the Blaschke condition $\sum_{n\geq 1}(1 - |\lambda_n|) < \infty$, then the infinite product*

$$B = \prod_{n\geq 1} b_{\lambda_n} = \lim_{r\to 1} \prod_{|\lambda_n|<r} b_{\lambda_n}$$

converges uniformly on compact subsets of $\mathbb{D}$ and even on compact subsets of $\mathbb{C}\setminus clos\{\frac{1}{\bar{\lambda}_n}\}_{n\geq 1}$. Moreover, $|B| \leq 1$ in $\mathbb{D}$, $|B| = 1$ a.e. on $\mathbb{T}$, and $Z(B) = (\lambda_n)_{n\geq 1}$ (counting multiplicity).

Proof. Set $B^r = \prod_{|\lambda_n|<r} b_{\lambda_n}$. Then for $0 \leq r < R < 1$, we have

$$\begin{aligned} \|B^R - B^r\|_2^2 &= 2 - 2\ \mathrm{Re}(B^R, B^r) \\ &= 2 - 2\ \mathrm{Re} \int B^R \bar{B}^r dm \\ &= 2 - 2\ \mathrm{Re} \int \frac{B^R}{B^r} dm \text{ (because} |B^r| = 1 \text{ on} \mathbb{T}). \end{aligned}$$

So by MVT for holomorphic function $\frac{B^R}{B^r}$ we get

$$\|B^R - B^r\|_2^2 = 2 - 2\ \mathrm{Re}\Big(\frac{B^R}{B^r}\Big)(0) = 2 - 2 \prod_{r\leq|\lambda_n|<R} |\lambda_n|.$$

By Blaschke condition $\sum_{n\geq 1} \log|\lambda_n|^{-1} < \infty$, the product

$$\prod_{n\geq 1} |\lambda_n|$$

converges, which implies $\lim_{r\to 1} \prod_{r\leq|\lambda_n|<R} |\lambda_n| = 1$. This shows that (B^r) is a Cauchy sequence in $H^2 \subset L^2$ for every $r = r_k \to 1$. So we deduce the existence of $B = \lim_{r\to 1} B^r$. Moreover, $|B| = 1$ a.e. on $\mathbb{T}$ because $|B^r| = 1$ on $\mathbb{T}$, and $B \in H^2$. As the point evaluation is continuous linear functional on H^2, the limit $\lim_{r\to 1} B^r(\lambda) = B(\lambda)$ exists uniformly on compact subsets of $\mathbb{D}$, and hence $|B(\lambda)| \leq 1$, $\lambda \in \mathbb{D}$. Using $\frac{B}{B^r} \to 1$ in H^2 (easy to see),

we get $\frac{B}{B^r} \to 1$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$ and

$$\lim_{r\to 1}\Big(\frac{B}{B^r}\Big)(\lambda) = 1. \tag{3.6}$$

This shows that $B(\lambda) = 0$, $|\lambda| < 1$ if and only if $\lambda = \lambda_n$ for some $n \geq 1$ (counting multiplicity). If $\lambda \neq \lambda_n$ and $B(\lambda) = 0$, then (3.6) will fail.

In order to prove convergence on compact subsets of $\mathbb{C} \setminus \mathrm{clos}\{\frac{1}{\bar\lambda_n}\}_{n\geq 1}$, the following observation is enough.

$$|b_{\lambda_n} - 1| = \frac{(|1-|\lambda_n|)(\lambda_n + |\lambda_n|z)}{\lambda(1-\bar\lambda z)} \leq \frac{(1-|\lambda_n|)(1+|z|)}{|\lambda_n|\big|z - \frac{1}{\bar\lambda_n}\big|} \leq c\frac{1-|\lambda|}{\mathrm{dist}(z,N)},$$

where $N = \mathrm{clos}\{\frac{1}{\bar\lambda_n} : n \geq 1\}$. $\square$

Corollary 3.15. *Let $f \in H^p(\mathbb{D})$, $p > 0$ with corresponding zero sequence $(\lambda_n)_{n\geq 1}$. Then there exists $g \in H^p(\mathbb{D})$ with $g(\xi) \neq 0, \forall\, \xi \in \mathbb{D}$ such that $f = Bg$ and $\|f\|_p = \|g\|_p$ on $L^p(\mathbb{T})$.*

This may be thought as the Blaschke filtering of the holomorphic functions.

Proof. Take $B^r = \prod_{|\lambda_n|<r} b_{\lambda_n}$, $0 < r < 1$. Clearly, $\frac{f}{B^r} \in$Hol$(\mathbb{D})$ and for $\rho \to 1$, we get $|B^r(\rho\xi)| \to 1$ uniformly on $\mathbb{T}$. Hence,

$$\Big|\Big|\frac{f}{B^r}\Big|\Big|_p^p = \lim_{\rho\to 1}\int_{\mathbb{T}}\Big|\frac{f}{B^r}(\rho\xi)\Big|^p dm(\xi) = \|f\|_p^p.$$

And thus by definition of $H^p(\mathbb{D})$,

$$\Big(\int_{\mathbb{T}}\Big|\frac{f}{B^r}(\rho\xi)\Big|^p dm(\xi)\Big)^{\frac{1}{p}} \leq \|f\|_p \text{ for every } 0 \leq \rho < 1.$$

Fix ρ, set $g = \frac{f}{B}$, and letting $r \to 1$, we obtain

$$\Big(\int_{\mathbb{T}}\Big|g(\rho\xi)\Big|^p dm(\xi)\Big)^{\frac{1}{p}} \leq \|f\|_p,$$

and hence $\|g\|_p \leq \|f\|_p$. The other inequality follows from $g = \frac{f}{B}$. $\square$

Question 3.16. * Is it possible to replace $\log|\cdot|$ in Jensen's inequality with some suitable increasing function?

Remark 3.17. It is useful to introduce the notion of the zero divisor (or multiplicity function) of a holomorphic function. For $f \in \mathrm{Hol}(\Omega)$, $\Omega \subset \mathbb{C}$, $f \not\equiv 0$, $\lambda \in \Omega$, set

$$d_f(\lambda) = \begin{cases} 0 & \text{if} f(\lambda) \neq 0 \\ m & \text{if} f(\lambda) = \cdots = f^{(m-1)}(\lambda) = 0 \text{ and} f^m(\lambda) \neq 0. \end{cases}$$

The value of $d_f(\lambda)$ is called zero multiplicity of λ. We can redefine the Blaschke condition. The zero divisor of $f \in \mathrm{Hol}(\mathbb{D})$ verifies the Blaschke condition if and only if

$$\sum_{\lambda\in\mathbb{D}} d_f(\lambda)(1-|\lambda|) < \infty.$$

The corresponding Blaschke product is given by

$$\prod_{\lambda\in\mathbb{D}} b_\lambda^{d_f(\lambda)} = \prod_{n\geq 1} b_{\lambda_n}^{d_f(\lambda_n)}.$$

Corollary 3.18. *Let $f \in H^p, p > 0$ then there exists $f_k \in H^p\,; k = 1, 2$ such that $f = f_1 + f_2$, $\|f_k\|_p \leq \|f\|_p$, and $f_k(z) \neq 0$ for $z \in \mathbb{D}$*

Indeed, we have $f = Bg$, and set $f_1 = \frac{1}{2}(1-B)g$, $f_2 = \frac{1}{2}(1+B)g$.

3.9. **Non-tangential boundary limits.** Recall that we have identified boundary limit $\tilde{f}$ of $f \in H^p(\mathbb{D})$ via

$$\lim_{r\to 1} \|f_r - \tilde{f}\|_p = 0,\ \tilde{f} \in H^p,\ 1 \leq p < \infty.$$

We shall see another convergence of $f(z)$ to its boundary values, namely the so-called non-tangential convergence a.e. on $\mathbb{T}$ for $f \in H^p(\mathbb{D})$ with $0 < p \leq \infty$.

Let μ be a complex valued Borel measure on $\mathbb{T}$ and $\mu \in \mathcal{M}(\mathbb{T})$. Let $d\mu = hdm + d\mu_s$, $h \in L^1(m)$ be Lebesgue decomposition of μ with respect to m. Then the derivative of μ with respect to m exists at almost every point $\xi \in \mathbb{T}$, in the following sense.

$$\lim_{\Delta\to\xi,\,\xi\in\Delta} \frac{\mu(\Delta)}{m(\Delta)} = \frac{d\mu(\xi)}{dm}(= h(\xi)),$$

where Δ is an arc on $\mathbb{T}$ tending to ξ. Such a point will be called **Lebesgue point** of μ. Note that the Poisson kernel satisfies $P(re^{it}) = \dfrac{1-r^2}{|1-re^{it}|^2}$. For $f \in L^p(\mathbb{T})\,(1 \leq p < \infty)$, we have

$$\begin{aligned} P_r * f(e^{it}) &= \int_{\mathbb{T}} \frac{1-r^2}{|1-re^{i(t-s)}|^2} f(e^{is})dm(e^{is}) \\ &= \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} f(\zeta)dm(\zeta)\ (\text{put} z = re^{it},\, \zeta = e^{is}) \\ &= f * P(z)\,(write). \end{aligned}$$

That is, $P_r * f(e^{it}) = f * P(z)$, where $z = re^{it}$. We see one of the most important result about non-tangential limit.

Theorem 3.19. (*P. Fatou's, 1996*) *Let $\mu \in \mathcal{M}(\mathbb{T})$ and $\xi \in \mathbb{T}$ be a Lebesgue point of μ, then Poisson integral of μ*

$$f(z) = P * \mu(z) = \int \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta)$$

has a non-tangential limit at ξ, and equal to $\frac{d\mu}{dm}(\xi)$. In other words, if S_ξ is an angular sector centered at ξ with bisector $[0, \xi]$ and opening angle less than π, then

$$\lim_{z \to \xi, z \in S_\xi} f(z) = \frac{d\mu}{dm}(\xi) \ a.e. \ on \ \mathbb{T}.$$

In particular, $\lim_{r \to 1} f(r\xi) = \dfrac{d\mu}{dm}(\xi)$ on $\mathbb{T}$. S_ξ is so-called Stolz angle at ξ.

Proof. We know that $S_\xi = \{z \in \mathbb{D} : |z - \xi| \le c(1 - |z|)\}$. Without loss of generality, we may suppose that $\mu(\mathbb{T}) = 0$ (just replace μ by $\mu - cm$), and by rotation, assume $\xi = 1$. Then there exists a left continuous function F on $[-\pi, \pi]$ of the bounded variation such that $\mu[e^{i\alpha}, e^{i\beta}) = F(\beta) - F(\alpha)$, and $F(-\pi) = F(\pi)$ (because $\mu(\mathbb{T}) = 0$). Writing $f(z) = \int_{-\pi}^{\pi} P(ze^{-it}) dF(t)$, $z \in \mathbb{D}$ and integrating by parts, we get

(3.7)
$$f(z) = -\int_{-\pi}^{\pi} \frac{dP(ze^{-it})}{dt} F(t) dt = -\int_{-\pi}^{\pi} k_z(t) \frac{F(t)}{t} dt = \int_{-\pi}^{\pi} k_z(t) 2\pi \frac{F(t) - F(-t)}{2t} \frac{dt}{2\pi},$$

where $k_z(t) = -t \frac{dP}{dt}(ze^{-it})$. Then the family $\{k_z\}_{z \in S_1}$ is a good kernel.

(i) Set $z = re^{i\tau}$, $|\tau| \le \pi$, $0 \le r < 1$. Then we have

$$\begin{aligned} ||k_z||_1 &= \int_{-\pi}^{\pi} \Big| t\frac{dP(ze^{-it})}{dt} \Big| dt \\ &\le \int_{\tau-\pi}^{\tau+\pi} (|\tau| + |s|) \Big| \frac{dP(ze^{-is})}{ds} \Big| ds, (\text{put } s = \tau - t) \\ &\le C_1 \frac{|\tau|}{1-r} + C_2 + \int_{-\pi}^{\pi} \Big| s\frac{dP(re^{-is})}{ds} \Big| ds \\ &= C_1 \frac{|\tau|}{1-r} + C_2 - \int_{-\pi}^{\pi} s\frac{dP(re^{-is})}{ds} ds \\ &\le C_1 \frac{|\tau|}{1-r} + C_2 - \frac{1}{2\pi}\left[sP(re^{-is})\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} sP(re^{-is}) ds \\ &\le \text{const}, \end{aligned}$$

as the quantity $\frac{|\tau|}{1-r}$ is uniformly bounded in S_1.

(ii) Again by partial integration, we get

$$\lim_{z\to 1, z\in S_1} \int_{-\pi}^{\pi} k_z(t) \frac{dt}{2\pi} = \lim_{z\to 1, z\in S_1} (1 - P(-re^{i\tau})) = 1.$$

(iii) By straightforward calculation, we get

$$k_z(t) = 2rt \frac{\sin(\tau - t)}{1 - 2r\cos(\tau - t) + r^2} P(re^{i(\tau - t)}) \to 0$$

uniformly on $\delta < |t| < \pi$, whenever $\delta > 0$.

From (3.7) and (ii), for fixed $\delta > 0$, as $z \to 1$, $z \in S_1$,

$$\begin{aligned} f(z) &= \frac{d\mu}{dm}(1) + \int_{-\pi}^{\pi} k_2(t) \Big(2\pi \frac{F(t) - F(-t)}{2t} - \frac{d\mu}{dm}(1) \Big) \frac{dt}{2\pi} + o(1) \\ &= \frac{d\mu}{dm}(1) + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} + \int_{-\delta}^{\delta} + o(1), \end{aligned}$$

where the latter integral is arbitrarily small for small δ (in view of (i) and

$$\Delta(t) = 2\pi \frac{F(t) - F(-t)}{2t} - \frac{d\mu}{dm}(1) = o(1)),$$

and the two former tend to zero as $z \to 1$ in S_1, for every fixed $\delta > 0$ (in view of (iii) and boundedness of $\Delta(t)$.) $\square$

Corollary 3.20. *If $f \in H^p(\mathbb{D})$, $0 < p \le \infty$, then the non-tangential boundary limits of f exist a.e. on $\mathbb{T}$. That is,*

$$\lim_{z\to\xi, z\in S_\xi} f(z) = \tilde{f}(\xi) \;\; \textit{for a.e.} \xi \in \mathbb{T}.$$

The boundary function $\xi \mapsto f(\xi)$ is in $L^p(\mathbb{T})$, and for $p \geq 1$, $f(\xi) = \tilde{f}(\xi)$ a.e. on $\mathbb{T}$.

Proof. For $p \geq 1$, the claim follows from (3.7) and the identification theorem (because radial limit exists). Note that for $f \in L^p\,(1 \leq p < \infty)$ and $d\mu = f dm$, we have

$$\begin{aligned} P * \mu(z) &= \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} f(\zeta) dm(\xi) \\ &= P_r * f(e^{it})\,(\text{let} z = r\xi) \\ &= f_{(r)}(e^{it}) = f(r\xi) \to f(\xi) \text{ as } r \to 1 \end{aligned}$$

in $L^p(\mathbb{T})$. Hence, there exists (r_k) such that $P * \mu(z) \to \tilde{f}(\xi)$ as $r_k \to 1$ for a.e. $\xi \in \mathbb{T}$. And by Fatou's theorem, $\lim_{z \to \xi, z \in S_\xi} f(z) = \tilde{f}(\xi)$ for a.e.$\xi \in \mathbb{T}$. Hence $f(\xi) = \tilde{f}(\xi)$ for a.e. $\xi \in \mathbb{T}$.

For general case, we know that $f = Bg = B(g^{1/p})^p$, where $g \in H^p(\mathbb{D})$. This implies $g^{1/p} \in H^1(\mathbb{D})$. The result follows from the previous reasoning. □

Notation: From now onward, we identify the functions $f \in H^p(\mathbb{D})$ with their boundary values on $\mathbb{T}$, and write $H^p(\mathbb{D}) = H^p(\mathbb{T})$, $0 < p \leq \infty$, where $H^p(\mathbb{T})$ is the collection of boundary functions of $H^p(\mathbb{D})$.

3.10. **The Riesz - Smirnov canonical factorization.** Here we see the main result of the Hardy space theory - a parametric representation of $f \in H^p$ as a product of Blaschke product, a singular inner function, an outer (maximal) function. The last two functions are exponential of integral depending on the holomorphic Schwarz - Herglotz kernel $z \to \frac{\zeta+z}{\zeta-z}$, whose real part is the Poisson kernel.

Theorem 3.21. *Let $f \in L^p$, $0 < p \leq \infty$ be such that $\log|f| \in L^1$, and define*

$$[f](z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|f(\zeta)| dm(\zeta)\right),\ |z| < 1.$$

Then

(i) $[f] \in H^p(\mathbb{D})$ *and* $|[f]| = |f|$ *a.e. on* $\mathbb{T}$.

(ii) If $0 \not\equiv g \in H^q(\mathbb{D})$, $q \geq 1$, *and* $|g| \leq |f|$ *a.e. on* $\mathbb{T}$, *then* $|g| \leq |[f]|$ *on* $\mathbb{D}$ *(and hence* $g \in H^p(\mathbb{D})$*).*

(iii) $\left[\frac{f}{g}\right] = \frac{[f]}{[g]}$ *and* $[[f]] = [f]$.

Proof. (i) Clearly, $[f]$ is a holomorphic function $(\mathbb{D})$. Recall that for a finite Borel measure μ and a convex function ψ, we have the Jensen-Young geometric mean inequality

$$\frac{\int \psi \circ F \, d\mu}{\int d\mu} \geq \psi\Big(\frac{\int F d\mu}{\int d\mu}\Big). \tag{3.8}$$

[Proof Let $F : (\Omega, \mu) \to I \subset \mathbb{R}$ (I is finite or infinite interval), set $\nu = \frac{\mu}{\int d\mu}$. Let $A = \{h : h(x) = ax + b;\ h \leq \psi \text{ on } I\}$. Then $h\big(\int F d\nu\big) = \int h \circ F d\nu \leq \int \psi \circ F d\nu$. We get the inequality since $\psi(x) = \sup\{h(x) :\ h \in A\}$.] By apply inequality (3.8) to the Borel measure $d\mu = \frac{1-|z|^2}{|\zeta - z|^2}\, dm(\zeta)$, we get

$$|[f]|^p = \exp\Big(\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} \log|f(\zeta)|^p dm(\zeta)\Big) \leq \int_{\mathbb{T}} |f(\zeta)|^p \frac{1-|z|^2}{|\zeta - z|^2} dm(\zeta).$$

Set $z = re^{it}$. By Fubini's theorem, we get

$$\int_0^{2\pi} \Big|[f](re^{it})\Big|^p \frac{dt}{2\pi} \leq \int_{\mathbb{T}} |f(\zeta)|^p \Big(\int_0^{2\pi} \frac{1-|z|^2}{|\zeta - z|^2} \frac{dt}{2\pi}\Big) dm(\zeta) = ||f||_p^p.$$

Now, by Fatou's theorem and its corollary there, we have

$$|[f](\xi)| = \lim_{r \to 1} \log |[f](r\xi)| = \log|f(\xi)| \text{ a.e.} \xi \text{ on } \mathbb{T}.$$

The modifications in the case $p = \infty$ are obvious.

(ii) Given that $0 \not\equiv g \in H^q(\mathbb{D})$, $q \geq 1$, and $|g| \leq |f|$ a.e. on $\mathbb{T}$. This implies $\log|g| \in L^1$, and hence by generalized Jenson's inequality (3.5), we get

$$\begin{aligned} \log|g(z)| &\leq \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} \log|g(\zeta)| dm(\zeta) \\ &\leq \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} \log|f(\zeta)| dm(\zeta) \\ &= \log|[f](z)|. \end{aligned}$$

(iii) is a direct consequence of the definition.

□

The following result ensures the existence of enough harmonic functions as Poisson integrals of finite Borel measures.

Theorem 3.22. (*G. Herglotz, 1911*) *Let u be a non-negative harmonic function on $\mathbb{D}$. Then there exists a unique finite Borel measure $\mu \geq 0$ such that $u = P * \mu$, that is*
$$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

Proof. By MVT we have for all z in $\mathbb{D}$
$$u_r(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} u_r(\zeta) dm(\zeta) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu_r(\zeta),$$
where we have set $u_r(z) = u(rz)$, $0 \leq r < 1$, and $d\mu_r = u_r dm$. Then μ_r is a positive measure and $\mathrm{Var}(\mu_r) = \mu_r(\mathbb{T}) = u_r(0) = u(0) < \infty$. Thus, the family $(u_r)_{0 \leq r < 1}$ is uniformly bounded in $\mathcal{M}(\mathbb{T})$, and has week* convergent subsequence μ_{r_n} that converges to $\mu \in \mathcal{M}(\mathbb{T})$. Recall that $\mathcal{M}(\mathbb{T})$ is dual of $C(\mathbb{T})^*$ with the duality $< f, \mu >= \int\limits_{\mathbb{T}} f d\mu$. Thus, if $f \in C(\mathbb{T})$, $f \geq 0$, then
$$\int_{\mathbb{T}} f d\mu = \lim_{n \to \infty} \int_{\mathbb{T}} f u_{r_n} dm \geq 0 \implies \mu \geq 0.$$
Moreover, since u is continuous on $\mathbb{D}$, for $z \in \mathbb{D}$, we have
$$u(z) = \lim_{n\to\infty} u(r_n z) = \lim_{n \to \infty} \int \frac{1-|z|^2}{|\zeta - z|^2} d\mu_{r_n}(\zeta) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta).$$
Uniqueness of μ: Note that $P * \mu(re^{it}) = \sum\limits_{n \in \mathbb{Z}} r^{|n|} \hat{\mu}(n) e^{int}$. For any ν such that $P * \mu = P * \nu$ implies $\hat{\mu}(n) = \hat{\nu}(n)$. Hence, $\mu = \nu$. □

Theorem 3.23. (*Singular inner function*)*: Let $S \in Hol(\mathbb{D})$, then the following are equivalent:*

(i) $|S(z)| \leq 1$ and $S(z) \neq 0$ on $\mathbb{D}$, $S(0) > 0$ and $|S(\xi)| = 1$ a.e. on $\mathbb{T}$.

(ii) there exists a unique finite Borel measure $\mu \geq 0$ on $\mathbb{T}$ with $\mu \perp m$ such that
$$S(z) = \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right), \ z \in \mathbb{D}.$$

Proof. (ii) implies (i) is a corollary of Fatou's theorem (because of $S \in H^\infty(\mathbb{D})$ by (ii)). For (i) implies (ii), let $u = \log |S|^{-1}$, then by Herglotz theorem, there exists μ such that
$$\log |S(z)|^{-1} = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta).$$
Once again by Fatou's theorem (and $|S(\xi)| = 1$ a.e. on $\mathbb{T}$), we get
$$\frac{d\mu}{dm}(\xi) = \lim_{r \to 1} u(r\xi) = 0 \ \text{ a.e. on} \mathbb{T}.$$
Hence $\mu \perp m$. □

Definition 3.24. A function S verifying (i) or (ii) of the preceding theorem is called a singular inner function.

Theorem 3.25. (*F. Riesz, V. Smirnov*) *Let $f \in H^p(\mathbb{D})$, $p > 0$. Then there exists a unique factorization $f = \lambda BS[f]$, where $\lambda \in \mathbb{C}$, $|\lambda| = 1$, B, S and $[f]$ are defined earlier.*

Proof. Set $g = \frac{f}{B}$. Then $|f| = |g|$ a.e. on $\mathbb{T}$, and hence $[g] = [f]$. Set $\lambda = \frac{g(0)}{[g](0)}$ and $S = \frac{g}{\lambda[g]}$. Then $f = Bg = B\lambda S[g] = \lambda BS[f]$. As B and $[f]$ are uniquely defined for f, the uniqueness of factorization follows. $\square$

Next, we consider the structure of the outer functions in H^p.

Theorem 3.26. (***Structure of outer function***) *Let $p, q, r \geq 1$ and $f \in H^p$. Then the following are equivalent.*

(i) There exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $f = \lambda[f]$.

(ii) for all $z \in \mathbb{D}$, the generalized Jensen inequality is equality:

$$\log|f(z)| = \int_{\mathbb{T}} P(z\bar{\xi}) \log|f(\xi)| dm(\xi). \tag{3.9}$$

(iii) Identity (3.9) holds for at least one $z \in \mathbb{D}$.

(iv) If $g \in H^q$ and $\frac{g}{f} \in L^r$, then $\frac{g}{f} \in H^r$ (integral maximal principle).

If $p = 2$, then (i)-(iv) are equivalent to

(v) the function f is outer in H^2(in the earlier sense i.e., $E_f = H^2$).

Proof. (i) implies (ii) is followed from the definition of $[f]$. The implication (iii) goes to (ii) is trivial. For (iii) implies (i), suppose (3.9) holds for some $z_o \in \mathbb{D}$. By Riesz-Smirnov factorization theorem, we have $f = \lambda BS[f]$, and by (3.9), we get

$$|f(z_o)| = |\lambda B(z_o)S(z_o)[f](z_o)| \implies |B(z_o)S(z_o)| = 1 \implies |B(z_o)| = |S(z_o)| = 1.$$

By maximum principle, $B = S =$constant$= 1$ in $\mathbb{D}$, implies $f = \lambda[f]$.

(i) implies (iv): If $g \in H^q$, then $g = \lambda_1 BS[g]$ and we get $\frac{g}{f} = \frac{\lambda_1 BS[g]}{(\lambda[f])} = \left(\frac{\lambda_1}{\lambda}\right) BS\left[\frac{g}{f}\right] \in H^r$ in view of Riesz-Smirnov theorem and by the hypothesis that $g/f \in L^r$.

(iv) $\implies$ (i): Let $f = \lambda BS[f]$ and set $g = \min(|f|, 1)$. Then $[g] \in H^\infty$ and $\left|\frac{[g]}{f}\right| \leq 1$ a.e. on $\mathbb{T}$. By (iv) we get $\frac{[g]}{f} \in H^r$ (r arbitrary). Again, we have $\frac{[g]}{f} = \lambda_1 B_1 S_1 \left[\frac{g}{f}\right] = \lambda_1 B_1 S_1 \frac{[g]}{[f]}$ (because $[[g]] = [g]$ and $\left[\frac{g}{f}\right] = \frac{[g]}{[f]}$), we get $1 \equiv \lambda\lambda_1 BB_1 SS_1 = \lambda_2 B_2 S_2$ with $|\lambda_2| = 1$, where B_2 is a Blaschke product and S_2 is a singular inner function. As $|B_2(z)| \leq 1$ and $|S_2(z)| \leq 1$ for all $z \in \mathbb{D}$, we get $|B_2| = |S_2| \equiv 1$ and hence $B_2 \equiv S_2 \equiv 1$. Thus, we conclude that $B = S = 1$, implies $f = \lambda[f]$.

It remains to show that (iv) and (v) are equivalent if $p = 2$. As (i)-(iii) are independent of choice of q and r, we get equivalence between (iv) as well with $p = 2$, and arbitrary q, r and with $p = q = r = 2$, (iv) is just earlier characterization of the outer function on H^2. □

Definition 3.27. Let $f \in H^p$, $p > 0$. The function $[f]$ is called the outer part of f, and λBS is called the inner part of f.

Notation: We write $[f] = f_{out}$ and $\lambda BS = f_{inn}$. If $f = \lambda[f]$, then f is called outer.

It is clear from the above theorem that if $p = 2$, then definition of inner and outer functions coincide with previous ones.

Corollary 3.28. *Let $w \in L^1_+(\mathbb{T})$, and $p \geq 1$. The followings are equivalent.*

(i) There exists $f \in H^p$, $f \not\equiv 0$ such that $|f|^p = w$ a.e. on $\mathbb{T}$.

(ii) $\log w \in L^1$.

Proof. As $H^p \subset H^1$, (i) implies (ii) follows from the boundary uniqueness theorem and (ii) implies (i) follows by taking $f = [w^{1/p}]$. Since

$$f(z) = \exp\left(\int_{\mathbb{T}} P(z\bar{\xi}) \log |w(\xi)|^{1/p} dm(\xi)\right),$$

by Theorem 3.5, $f \in H^p(\mathbb{D})$ because

$$|f(z)|^p = \exp\left(\int_{\mathbb{T}} P(z\bar{\xi}) \log |w(\xi)| dm(\xi)\right).$$

By Fatou's theorem, we get $|f|^p = w$ a.e. on $\mathbb{T}$. □

3.11. **Approximation by inner functions and Blaschke products.** Using Fatou's theorem, we prove two important theorems on uniform approximation by inner functions.

Theorem 3.29. (*R. Douglas and W. Rudin, 1969*) *Let Σ be the set of all inner functions. Then*

$$L^{\infty}(\mathbb{T}) = clos_{L^{\infty}}\left(\bar{\Theta}H^{\infty} : \Theta \in \Sigma\right) = \overline{span}_{L^{\infty}}\left(\bar{\Theta}_1\Theta_2 : \Theta_1, \Theta_2 \in \Sigma\right). \tag{3.10}$$

Moreover, any unimodular function in $L^{\infty}(\mathbb{T})$ belongs to $clos_{L^{\infty}}(\Pi)\left(\bar{\Theta}_1\Theta_2 : \Theta_1, \Theta_2 \in \Sigma\right)$.

Proof. It is enough to show that $\chi_\sigma \in \overline{\text{span}}_{L^{\infty}}\left(\bar{\Theta}_1\Theta_2 : \Theta_1, \Theta_2 \in \Sigma\right)$ for every Borel measurable set σ in $\mathbb{T}$. Let

$$f_n = \left[n\chi_\sigma + \frac{1}{n}\chi_{\mathbb{T}\setminus\sigma}\right],\ n = 2, 3, \dots$$

and $A_n = \{z \in \mathbb{C} : \frac{1}{n} < |z| < n\}$. It is clear that $f_n(\mathbb{D}) \subset A_n$ (by maximum principle) and $f_n(\mathbb{T}) \subset \partial A_n$. Now, let $\phi_1(\zeta) = \zeta + \frac{1}{\zeta}$ for $\zeta \in \mathbb{C} \setminus \{0\}$, and $w : \phi_1(A_n) \to \mathbb{D}$ be a conformal (Riemann) mapping of the ellipse $\phi_1(A_n)$ onto $\mathbb{D}$. Since the boundary of ellipse is smooth, w can be continuously extended to $\operatorname{clos}\phi_1(A_n)$, and hence

$$w \circ \phi_1 \circ f_n = \theta_1$$

is an inner function (because $\theta_1 \in H^{\infty}(\mathbb{D})$, and by Fatou's theorem $|\theta_1| = 1$ a.e. on $\mathbb{T}$). Since w^{-1} is continuous on $\operatorname{clos}(\mathbb{D})$, it can be approximated by its Fejer polynomials. Therefore,

$$f_n + \frac{1}{f_n} = \phi_1 \circ f_n = w^{-1} \circ \theta_1 \in \text{span}_{L^{\infty}}\left(\theta_1^n : n \geq 0\right).$$

Doing the same for the function $\phi_2(\zeta) = \zeta - \frac{1}{\zeta}$, we get an inner function θ_2 such that $f_n - \frac{1}{f_n} \in \overline{\text{span}}_{L^{\infty}}\left(\theta_2^n : n \geq 0\right)$. Hence $f_n \in \overline{\text{span}}_{L^{\infty}}\{\theta_1^k\theta_2^n : k, n \geq 0\}$, implies

$$|f_n|^2 \in \overline{\text{span}}_{L^{\infty}}\left(\theta_1^k\theta_2^n\theta_1^{-l}\theta_2^{-m} : k, n, l, m \geq 0\right).$$

Thus,

$$\chi_\sigma + \frac{1}{n^4}\chi_{\mathbb{T}\setminus\sigma} \in \overline{\text{span}}_{L^{\infty}}\left(\bar{\Theta}_1\Theta_2 : \Theta_1, \Theta_2 \in \Sigma\right),\ \text{for} n = 1, 2, \dots.$$

Letting $n \to \infty$, we get $\chi_\sigma \in \overline{\text{span}}_{L^{\infty}}\left(\bar{\Theta}_1\Theta_2 : \Theta_1,\ \Theta_2 \in \Sigma\right)$.

Let $u \in L^{\infty}(\mathbb{T})$, and $|u| = 1$ a.e. and $u_1 \in L^{\infty}(\mathbb{T})$ with $|u_1| = 1$ a.e. and $u = u_1^2$. Given $\epsilon > 0$, by (3.10) there exists $\varphi, \Theta_j \in \Sigma$ such that $|u_1 - \bar{\varphi}g| < \epsilon$, where $g = \sum_{j=1}^{n} a_j\Theta_j$, $a_j \in \mathbb{C}$.

Set $\Theta = \prod_{j=1}^{n} \Theta_j$, and observe that $\bar{g}\Theta \in H^\infty$. Since $[\bar{g}\Theta] = [g]$ (because $|\bar{g}\Theta| = |\bar{g}|$), the inner-outer factorizations of g and $\bar{g}\Theta$ are of the form $\bar{g}\Theta = v[g]$ and $g = w[g]$, where $v, w \in \Sigma$, and $1 - \epsilon < |[g]| < 1 + \epsilon$. Now, $|\bar{u_1} - \varphi\bar{g}| = |\bar{u_1} - \varphi\bar{\Theta}v[g]| < \epsilon$ gives

$$\left| \frac{1}{\bar{u_1}} - \frac{1}{\phi\bar{\Theta}v[g]} \right| < \frac{\epsilon}{1-\epsilon}.$$

Since $|u_1 - a| < \epsilon$ and $|u_1 - b| < \epsilon$ implies that $|u_1^2 - ab| \leq |u_1 - a| + |a||u_1 - b|$, we obtain

$$\left| u - \bar{\phi}w[g]\bar{\phi}\Theta\bar{v}\frac{1}{[g]} \right| < \frac{2\epsilon}{1-\epsilon},$$

which completes the proof. □

Theorem 3.30. (*O. Frostman, 1935*) *Let* Θ *be a (non-constant) inner function and* $\zeta \in \mathbb{T}$. *Then* $b_{t\zeta} \circ \Theta$ *are Blaschke products with simple zeros for a.e.* $t \in (0,1)$, *where* $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$, $\lambda \in \mathbb{D}$. *In particular,* Θ *is a uniform limit of Blaschke products with simple zeros.*

Proof. Let $\zeta = 1$. Then $\tilde{\Theta}_t = b_t \circ \Theta$ is an inner function for all $t \in [0,1)$ (by definition of b_t and Θ), and it has factorization $\tilde{\Theta}_t = \lambda BS[\tilde{\Theta}_t]$. Let μ_t be the singular measure corresponding S. Then by Jensen formula (and expression of S and $S \in H^\infty$ with $\|S\|_\infty \leq 1$), we get

$$\mu_t(\mathbb{T}) = \log|S(0)|^{-1} = \int_{\mathbb{T}} \log|S(r\xi)|^{-1} dm(\xi) \leq \int_{\mathbb{T}} \log|\tilde{\Theta}_t(r\xi))|^{-1} dm(\xi) = g(r,t),$$

for all $t, r \in [0,1)$. Therefore, it is sufficient to check that $\lim\limits_{r\to 1} g(r,t) = 0$ a.e. $t \in (0,1)$. Note that $r \mapsto \int_{\mathbb{T}} \log|f(r\xi)| dm(\xi)$ increases with $r \nearrow 1$, hence $r \mapsto g(r,t)$ decreases for every t. Secondly, $t \longmapsto g(0,t) = \log|\tilde{\Theta}_t(0))|^{-1}$ is integrable on $[0,1)$. By dominated convergence theorem, we get

$$\int_0^1 \lim_{r\to 1} g(r,t)dt = \lim_{r \to 1} \int_{\mathbb{T}} \left(\int_0^1 \log|b_t \circ \Theta(r\xi)|^{-1} dt \right) dm(\xi).$$

The last limit is zero since the function

$$u : w \mapsto \int_0^1 \log|b_t(w)|^{-1} dt$$

is well defined and continuous in $\bar{\mathbb{D}}$ and $u(w) = 0$ for $|w| = 1$ (see below). Hence $\mu_t(\mathbb{T}) = 0$ for a.e. $t \in [0,1)$. The same reasoning for any other $\zeta \in \mathbb{T}$. The zeros of $b_\lambda \circ \Theta$ are simple if $\lambda - \Theta(z_j) \neq 0$, $\forall\, j$, where $(z_j)_{j\geq 1}$ are the zeroes of Θ'. Indeed, if $b_\lambda(\Theta(z)) = 0$, then $\lambda - \Theta(z_j) = 0$ and hence $(b_\lambda \circ \Theta)'(z) = b'_\lambda(\Theta(z))\Theta'(z) \neq 0$.

Finally, we show thar u is continuous on $\bar{\mathbb{D}}$. Note that the integrals $\int_0^1 \log|1-tw|dt$ and $\int_0^1 \log|t-w|dt$ are similar and for $w = x+iy$, we have

$$\int_0^1 \log|t-w|^2 dt = \int_0^1 \log\{(t-x)^2 + y^2\}dt$$

is continuous in x and y (for instance $\int_0^1 \log(t-x)^2 dt = \chi_{(0,1)} * \log(x^2)$).

$\square$

4. Szegö infimum and generalized Phragmen-Lindelöf principle

In this section, we consider two applications of the canonical Riesz-Smirnov factorization. Namely, the Szegö infimum $\mathrm{dist}(1, H^2_0(\mu))$ is expressed in terms of measure μ, the cyclic functions of $L^2(\mathbb{T})$ are described. The classical logarithmic integral criterion for completeness of the polynomials, the case of incompleteness, and the closure of the polynomials $H^2(\mu)$ is described in terms of the outer function related to Radon-Nikodym derivative $w = \frac{d\mu}{dm}$. We consider outer functions, their extremal and extension properties, and distribution value properties. The important Smirnov subclass of Nevanlinna functions is considered. After transferring these results to an arbitrary simply connected domain of $\mathbb{C}$, we use these techniques to get a remarkably general Phragmen-Lindelöf type principle due to Smirnov (1920) and then by Helson (1960).

4.1. Szegö infimum and weighted polynomial approximation.

Theorem 4.1. (*Szegö, Kolmogorov*) *Let $d\mu = wdm + d\mu_s$ be a Borel measure. Then*
$$\inf_{p\in\mathbb{P}^0_+} \int_{\mathbb{T}} |1-p|^2 d\mu = \exp\left(\int_{\mathbb{T}} \log w\, dm\right).$$

Proof. We know that the infimum is equal to $|\hat{f}(0)|^2$ if there exists an outer function f such that $|f|^2 = w$ and otherwise 0. On the other hand, (by Corollary 3.10 and Theorem 3.21) such an outer function exists if and only if $\log w \in L^1$. In this case, we have
$$f(z) = \exp\Big(\int_{\mathbb{T}} \frac{\xi+z}{\xi-z} \log w^{\frac{1}{2}} dm(\xi)\Big)$$
and $|\hat{f}(0)|^2 = |f(0)|^2 = \exp(\int_{\mathbb{T}} \log wdm)$. □

Let $f \in L^2(\mathbb{T})$, and write $E_f = \overline{\mathrm{span}}\{z^n f : n \geq 0\}$. If $E_f = L^2(\mathbb{T})$, we say f is a cyclic vector. Note that the half of the trigonometric system $(z^n)_{n\geq 0}$ is far from being complete in $L^2(\mathbb{T})$, but multiplying by a suitable function f one can get completeness property i.e. $\overline{\mathrm{span}}\{z^n f : n \geq 0\} = L^2(\mathbb{T})$. It may happen that for different halfs of $(z^n)_{n\in\mathbb{Z}}$, nothing similar is true.

Corollary 4.2. *Let $f \in L^2$. Then $E_f = \overline{span}\{z^n f : n \geq 0\} = L^2$ if and only if $f(\xi) \neq 0$ a.e. on $\mathbb{T}$ and $\int_{\mathbb{T}} \log|f| dm = -\infty$.*

By Wiener Theorem 1.4 we have $zE_f = E_f$ if and only if $E_f = \chi_\sigma(L^2(\mathbb{T})$, and by Beurling- Helson Theorem 1.5, $zE_f \subset E_f$, there exists $\theta \in E_f$ such that $E_f = \theta H^2$ ($|\theta| = 1\, a.e.$). Thus, E_f is reducing if and only if there does not exist $g \in H^2$ such that $|g| = |f|$ a.e. $\mathbb{T}$. That is, if and only if $\log|f| \notin L^1$. We easily deduce the necessity of the condition claimed.

For the sufficiency, we again use Theorem 1.4 to get $E_f = \chi_\sigma L^2(\mathbb{T})$. As $f \in \chi_\sigma L^2(\mathbb{T})$ and $f \neq 0$ a.e. on $\mathbb{T}$ we get $\sigma = \mathbb{T}$.

Example 4.3. (a) If $f(e^{i\theta}) = |1 - e^{i\theta}|^\alpha$, $\alpha > -\frac{1}{2}$, then $E_f \neq L^2(\mathbb{T})$.
(b) If $f(e^{i\theta}) = \exp\left(\frac{-1}{1-e^{i\theta}}\right)$, then $E_f = L^2(\mathbb{T})$.

The following two theorems are final statements on weighted polynomial approximation on the circle $\mathbb{T}$.

Theorem 4.4. *Let μ be a positive measure on $\mathbb{T}$ and let $w = \frac{d\mu}{dm}$ its Radon-Nikodym derivative. Then polynomials $\mathbb{P}_+$ are dense in $L^2(\mu)$ if and only if $\log w \notin L^1(\mathbb{T})$.*

Proof. This is immediate from Corollary 2.4 and Theorem 4.1. □

Theorem 4.5. *Let μ be a positive measure on $\mathbb{T}$, let $d\mu = wdm + d\mu_s$ be its Lebesgue decomposition and suppose that $\log w \in L^1(\mathbb{T})$. Let $\phi \in H^2$ be the outer function defined by $\phi = [w^{\frac{1}{2}}]$. Then closure $H^2(\mu) = clos_{L^2(\mu)}\mathbb{P}_+$ is given by*

$$H^2(\mu) = L^2(\mu_s) \oplus (\phi^{-1}H^2) = L^2(\mu_s) \oplus \{f \in \ Hol(\mathbb{D}) : \ f\phi \in H^2\}.$$

Proof. Indeed, Corollary 2.1 gives $H^2(\mu) = H^2(wdm) \oplus L^2(\mu_s)$ and Lemma 2.3 and Theorem 4.1 show that $H^2(wdm)$ is 1-invariant (non-reducing) subspace of $L^2(wdm)$ (see also Remark 2.2). Now, Theorem 1.8 implies that $H^2(wdm) = \phi^{-1}H^2$. □

4.2. **How to recognize an outer function.** It is of practical importance to know how to recognize an outer function. We recognize outer function in terms of its boundary behavior.

Fact 4.6. If $f \in H^p(\mathbb{D})$, $p \geq 1$ and $\inf_{z\in\mathbb{D}} |f(z)| > 0$, then f is outer

It is clear that for $g \in H^q\,(q \geq 1)$ we have $\frac{g}{f} \in H^q$ and hence by Theorem 3.26 f is outer.

Exercise 4.7. What is an interpretation of the above fact in case of $p = 2$ on the circle.

Theorem 4.8. *(V. Smirnov, 1928) If $f \in Hol(\mathbb{D})$ and $Ref(z) \geq 0$ for all $z \in \mathbb{D}$, then $f \in H^p$, $0 < p < 1$, and f is outer.*

Proof. By hypothesis, $z \longmapsto (f(z))^p$ is analytic and we can choose $\arg f(z)$ such that $|\arg f(z)^p| \leq p\pi/2$, $z \in \mathbb{D}$. Hence, if $0 < p < 1$, then there exists $c_p > 0$ such that $|f(z)|^p \leq c_p\ \mathrm{Re}f(z)^p$. The MVT applied to the harmonic function $\mathrm{Re}f(z)^p$ gives
$$\int_0^{2\pi} |f(re^i\theta)|^p \frac{dt}{2\pi} \leq c_p \int_0^{2\pi} \mathrm{Re}(f(re^{i\theta})^p)\frac{dt}{2\pi} = c_p\ \mathrm{Re}(f(0)^p),$$
For $0 \leq r < 1$ and hence $f \in H^p$, $0 < p < 1$. Moreover, since $\mathrm{Re}(\frac{1}{f(z)}) \geq 0$ in $\mathbb{D}$, we have f and $\frac{1}{f}$ in H^p, $0 < p < 1$. By Fact 4.10 (below), f is an outer function. $\square$

Example 4.9. (V. Smirnov,1928) Let $\mu \in \mathcal{M}(\mathbb{T})$ and set
$$f_\mu(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu(\xi).$$
Then

(i) $f_\mu \in H^p$, $0 < p < 1$.
(ii) If $\mu \geq 0$, then f_μ is outer.

Fact 4.10. Let $f_1,\ f_2 \in H^p$. Then $f_1 f_2$ is outer if and only if f_1, f_2 are outer. In particular, if $f \in H^p$ and $\frac{1}{f} \in H^q\,(p, q > 0)$, then f is outer.

This follows by the uniqueness of the factorization Theorem 3.25.

Example 4.11. If a polynomial p has no zero in the open disc $\mathbb{D}$, then p is outer. Consider $p(z) = \text{constt}\prod_{i=1}^{n}\Big(1-\dfrac{z}{\xi_i}\Big)$, $|\xi_i| \geq 1$. As $|z| < 1$ and $|\xi_i| \geq 1$, we have $\text{Re}\big(1\text{-}\frac{z}{\xi_i}\big) \geq 0$. By applying Theorem 4.8 and Fact 4.10.

Fact 4.12. Let f be an outer function and let $h \in H^p\,(p \geq 1)$. If $|f| \leq |h|$ on $\mathbb{D}$, then h is outer.

Obviously, $\frac{f}{h} \in H^\infty$ and $\frac{f}{h}$ has no zeros in $\mathbb{D}$. By Theorem 3.25 we get the representation $\frac{f}{h} = \lambda SF$, where F is outer. Suppose that h is not outer. Then $h = \lambda_1 S_1 F_1$ with S_1 is a non-trivial singular inner function and $f = (\lambda\lambda_1)(SS_1)(FF_1)$ with $SS_1 \not\equiv$ constt, which contradicts the hypothesis.

Theorem 4.13. *Let $p > 0$.*

(i) Let $f_n \in H^p$ be a sequence of outer functions with $f_n(0) > 0$. If $|f_n| \searrow$ on $\mathbb{T}$, then $f(z) = \lim_{n\to\infty} f_n(z)$, $z \in \mathbb{D}$ exists uniformly on compact sets. Moreover, if $\lim_{n\to\infty} f_n(0) = 0$, then $f \equiv 0$, otherwise f is an outer H^p function.

(ii) Let $f \in H^p$ be an outer function. Then there exists a sequence of outer functions $f_n \in H^p$ and $\inf_{z\in\mathbb{D}} |f_n(z)| > 0$, $n \geq 1$, $|f_n| \searrow |f|$ on $\mathbb{T}$ (and hence on $\mathbb{D}$) and $f(z) = \lim_{n\to\infty} f_n(z)$, $z \in \mathbb{D}$.

Proof. (i) As the functions f_n are outer, we have

$$\log|f_n(z)| = \int_{\mathbb{T}} P(z\bar{\xi})\log|f_n(\xi)|dm(\xi).$$

To show the uniform convergence of f_n, it is enough to show that f_n is uniformly Cauchy sequence. For this, we will show $\log|f_n(z)|$ is a uniformly Cauchy.

$$\begin{aligned}
|\log|f_n(z)| - \log|f_{n+p}(z)|| &= \Big|\int_{\mathbb{T}} P(z\bar{\xi})\log\frac{|f_n(\xi)|}{|f_{n+p}(\xi)|}dm(\xi)\Big| \\
&\leq \sup_{|z|\leq R}|P(z\bar{\xi})|\int_{\mathbb{T}}\Big|\log\frac{|f_n(\xi)|}{|f_{n+p}(\xi)|}\Big|dm(\xi) \\
&= \text{const}\int_{\mathbb{T}}\log\frac{|f_n(\xi)|}{|f_{n+p}(\xi)|}dm(\xi) \\
&= \text{const}\Big(\int_{\mathbb{T}}\log|f_n(\xi)|dm(\xi) - \int_{\mathbb{T}}\log|f_{n+p}(\xi)|dm(\xi)\Big).
\end{aligned}$$

The conclusion is followed by monotone convergence theorem.

Suppose that $\inf\limits_{n\geq 1} f_n(0)=0$, then
$$\lim_{n\to\infty}\int_{\mathbb{T}}\log|f_n|dm=\lim_{n\to\infty}\log f_n=-\infty.$$
For a point $z_0\in\mathbb{D}$, we have $P(z_0\bar{\xi})\leq\frac{1+|z_0|}{1-|z_0|}=C_0$. Hence,
$$\log|f_n(z_0)|\leq C_0\int_{\mathbb{T}}\log|f_n|dm.$$
We conclude that $\lim\limits_{n\to\infty}\log|f_n(z_0)|=-\infty$ and similarly for all $z\in\mathbb{D}$ and we get $f\equiv 0$.

If $\inf\limits_{n\geq 1} f_n(0)>0$ and $|f_n|\searrow h$ on $\mathbb{T}$, then
$$\int_{\mathbb{T}}\log h dm=\lim_{n\to\infty}\int_{\mathbb{T}}\log|f_n|dm>-\infty,$$
and hence $\log h\in L^1$. Now, it is obvious that $\lim\limits_{n\to\infty} f_n(z)=f(z)$ with $f=[h]$.

(ii) Without loss of generality, we may assume that $f(0)>0$. Set $f_n=[|f|+\delta_n]$, where $\delta_n>0$ an appropriate sequence with $\lim\limits_{n\to\infty}\delta_n=0$ and $\int_{\mathbb{T}}\log(|f|+\delta_n)dm<\infty$. Then f_n satisfies the desired properties.

$\square$

4.3. **The Smirnov class $\mathcal{D}$.** We know that Nevanlinna class can be represented as
$$\mathrm{Nev}=\Big\{f\in \mathrm{Hol}(\mathbb{D}):\ \text{there exist } f_1,f_2\in\bigcup_{p>0}H^p \text{ such that } f=f_1/f_2\Big\}$$
and let
$$\mathcal{D}=\Big\{f\in \mathrm{Hol}(\mathbb{D}):\ \text{there exist } f_1,f_2\in\bigcup_{p>0}H^p \text{ such that } f=f_1/f_2 \text{ and } f_2 \text{ is outer}\Big\}$$
be the **Smirnov class** (sometimes denoted by Nev_+).

Lemma 4.14. *We have*
$$Nev=\big\{f\in\ Hol(\mathbb{D}):\ \textit{there exist } f_1,f_2\in H^\infty \textit{ such that } f=f_1/f_2\big\} \textit{ and}$$
$$\mathcal{D}=\big\{f\in\ Hol(\mathbb{D}):\ \textit{there exist } f_1,f_2\in H^\infty \textit{ such that } f=f_1/f_2 \textit{ and } f_2 \textit{ is outer}\big\}.$$

Proof. Let $f\in$Nev, $f\not\equiv 0$ and $f=\frac{f_1}{f_2}$, where $f_1,f_2\in H^1$ have canonical factorizations $f_1=\lambda[f_1]B_1S_1$ and $f_2=\lambda[f_2]S_2$. Set $F_1=\lambda[\min(1,|f|)]B_1S_1$ and $F_2=[\min(|f|^{-1},1)]S_2$. Clearly $F_1,F_2\in H^\infty$ and since $|f|.\min(|f|^{-1},1)=\min(1,|f|)$, we also get $f=\frac{F_1}{F_2}$.

□

4.4. **The Smirnov class $\mathcal{D}$.** We know that Nevanlinna class can be represented as

$$\mathrm{Nev} = \Big\{ f \in \ \mathrm{Hol}(\mathbb{D}) : \ \text{there exist } f_1, f_2 \in \bigcup_{p>0} H^p \text{ such that } f = f_1/f_2 \Big\}$$

and let

$$\mathcal{D} = \Big\{ f \in \ \mathrm{Hol}(\mathbb{D}) : \ \text{there exist } f_1, f_2 \in \bigcup_{p>0} H^p \text{ such that } f = f_1/f_2 \ \text{ and } f_2 \text{ is outer} \Big\}$$

be the **Smirnov class** (sometimes denoted by Nev_+).

Lemma 4.15. *We have*

$$\mathit{Nev} = \big\{ f \in \ \mathit{Hol}(\mathbb{D}) : \ \textit{there exist } f_1, f_2 \in H^\infty \textit{ such that } f = f_1/f_2 \big\} \textit{ and}$$

$$\mathcal{D} = \big\{ f \in \ \mathit{Hol}(\mathbb{D}) : \ \textit{there exist } f_1, f_2 \in H^\infty \textit{ such that } f = f_1/f_2 \textit{ and } f_2 \textit{ is outer} \big\}.$$

Proof. Let $f \in$Nev, $f \not\equiv 0$ and $f = \frac{f_1}{f_2}$, where $f_1, f_2 \in H^1$ have canonical factorizations $f_1 = \lambda[f_1]B_1S_1$ and $f_2 = \lambda[f_2]S_2$. Set $F_1 = \lambda[\min(1, |f|)]B_1S_1$ and $F_2 = [\min(|f|^{-1}, 1)]S_2$. Clearly $F_1, F_2 \in H^\infty$ and since $|f|. \min(|f|^{-1}, 1) = \min(1, |f|)$, we also get $f = \frac{F_1}{F_2}$.

□

Definition 4.16. A function $f \in$ Nev is called outer if there exist two outer functions f_1, f_2 such that $f = \frac{f_1}{f_2}$.

Properties 4.17. (of the class $\mathcal{D}$ and Nevanlinna outer functions)

(a) If f is outer, then $f \in \mathcal{D}$.

(b) If f_1 and f_2 is outer, then so is $f_1 f_2$.

(c) If $f_1 f_2$ are outer, and $f_1, f_2 \in \mathcal{D}$, then f_1, f_2 are outer.

(d) If $f_1, f_2 \in \mathcal{D}$, then $f_1 f_2 \in \mathcal{D}$.

(e) If $F \in \mathrm{Hol}(\mathbb{D})$, $G \in \mathcal{D}$ and $|F| \le |G|$ in $\mathbb{D}$, then $F \in \mathcal{D}$.

To verify (c), just let $G = \frac{G_1}{G_2}$ with $G_1, G_2 \in H^\infty$, and G_2 outer. By hypothesis $|G_2 F| \le |G_1|$ in $\mathbb{D}$, and hence $G_2 F \in H^\infty$. We conclude that $F = \frac{G_2 F}{G_2} \in \mathcal{D}$

Theorem 4.18. (*Generalized Maximum Principle*) *Let $f \in \mathcal{D}$ and g be an outer function in Nev. If $|f| \leq |g|$ on $\mathbb{T}$, then $|f| \leq |g|$ on $\mathbb{D}$.*

Proof. Let $f = \frac{f_1}{f_2}$ and $g = \frac{g_1}{g_2}$ where f_2, g_1 and g_2 are outer functions in H^∞ and $f_1 \in H^\infty$. By assumption $|f_1 g_2| \leq |f_2 g_1|$ on $\mathbb{T}$ and hence $|f_1 g_2| \leq |[f_1 g_2]| \leq |[f_2 g_1]| = |f_2 g_1|$ in $\mathbb{D}$. □

Remark 4.19. This result is not true in general if $f \in \text{Nev}\backslash\mathcal{D}$ and/or if g is not outer.

4.5. **A conformably invariant framework.** Here we consider the classes Nev(Ω) and $\mathcal{D}(\Omega)$, where Ω is a simply connected domain ($\neq \mathbb{C}$), that is, domains that are conformably equivalent to the open unit $\mathbb{D}$.

Definition 4.20. Define

$$H^\infty(\Omega) = \{f \in \text{Hol}(\Omega) : \|f\|_{H^\infty} = \sup_{z\in\Omega} |f(z)| < \infty\}$$

and

$$\text{Nev}(\Omega) = \{f \in \text{Hol}(\Omega) : \text{ there exist } f_1, f_2 \in H^\infty(\Omega) \text{ such that } f = f_1/f_2\}.$$

For $\omega : \mathbb{D} \to \Omega$ be an onto conformal map. A function $f \in \text{Nev}(\Omega)$ is called outer if $f \circ w$ is an outer in Nev$(\mathbb{D})$. With this definition, we get
$\mathcal{D}(\Omega) = \big\{f \in \text{Hol}(\Omega) : \text{ there exist } f_1, f_2 \in H^\infty(\Omega) \text{ such that } f = f_1/f_2 \text{ and } f_2 \text{ is outer}\big\}$.
The following two results are simple factorization to Ω of the corresponding well known facts in $\mathbb{D}$. Note if $\omega : \Omega \to \mathbb{D}$ extends to a homeomorphism of clos (Ω) onto clos $(\mathbb{D})$, then we say Ω is **Jordan domain**.

Lemma 4.21. (*Generalized Maximum Principle*) *Let Ω be a Jordan domain. Let $\lambda \in \partial\Omega$, $f \in \mathcal{D}(\Omega) \cap C\big(\text{ clos}(\Omega)\backslash\{\lambda\}\big)$ and let g be an outer function such that $g \in C\big(\text{ clos}(\Omega)\backslash \{\lambda\}\big)$ and $|f| \leq |g|$ on $\partial\Omega \setminus \{\lambda\}$. Then $|f| \leq |g|$ on Ω.*

Lemma 4.22. *Let $f \in H^\infty(\Omega)$. Then f is outer if and only if there exists a sequence of outer functions $(f_n)_{n\geq 1} \in H^\infty(\Omega)$ such that*

$$\inf_{z\in\Omega} |f_n(z)| > 0,\ n \geq 1,\ \ \lim_{n\to\infty} f_n(z) = f(z),\ \ |f_n(z)| \searrow |f(z)|, z \in \Omega.$$

Corollary 4.23. *Let $\Omega_1 \subset \Omega_2$ be two simply connected domains and $f \in Nev(\Omega_2)$.*

(i) If f is outer on Ω_2, then $f|_{\Omega_1}$ is outer on Ω_1.

(ii) If $f \in \mathcal{D}(\Omega_2)$, then $f|_{\Omega_1} \in \mathcal{D}(\Omega_1)$.

4.6. **The generalized Fragmen-Lindlöf principle.** The result of Theorem 4.18 and Lemma 4.21 are, in fact, the versions of the Fragmen-Lindlöf principle. The difference is that, in general, the mejorants are not given by analytic functions.

Let Ω be a Jardon Domain, let M and M_* be two non-negative functions on Ω, and let $\omega \in C(\partial\Omega \setminus \{\lambda\})$, where $\lambda \in \partial\omega$, $\Omega > 0$. Then M_* is called Fragmen-Lindlöf majorant for M and ω if for every $f \in \mathrm{Hol}(\Omega) \cup C(\mathrm{clos}(\Omega) \setminus \{\lambda\})$ with $|f| \le M$ on $\partial\Omega \setminus \{\lambda\}$ we have $|f| \le M_*$.

Theorem 4.24. (*Generalized Fragmen-Lindlöf principle*) *Let $f \in \mathcal{D}(\Omega)$ and $G \in Nev(\Omega) \cap C(clos(\Omega) \setminus \{\lambda\})$ be such that $M \le |F|$ on Ω, $\omega \le |G|$ on $\partial\Omega \setminus \{\lambda\}$. Then either there exists an outer function $[\omega \circ \omega]$ (and then $M_* = |[\omega \circ \omega] \circ \omega^{-1}|$ is a Fragmen-Lindlöf majorant for M and ω) or $f \equiv 0$ for all $f \in Hol(\Omega) \cup C(clos(\Omega) \setminus \{\lambda\})$ such that $|f| \le M$ on Ω and $|f| \le \omega$ on $\partial\Omega\{\lambda\}$ (and then $M_* = 0$).*

Proof. In view of (e) of Properties 4.17, the inequalities $|F| \le M \le |F|$ show that $f \in (\Omega)$. If there exists $f \not\equiv 0$, $f \in \mathrm{Nev}(\Omega)$ such that

$$|f \circ \omega| \le \omega \circ \omega \le |G \circ \omega|$$

on $\mathbb{T} \setminus \omega^{-1}(\{\lambda\})$, then we can define the outer function $[\omega \circ \omega]$. Applying Lemma 4.21 we get $|f \circ \omega| \le |[\omega \circ \omega]|$ on $\mathbb{T} \setminus \omega^{-1}(\{\lambda\})$ and hence the desired result. $\square$

5. Harmonic analysis in $L^2(\mathbb{T}, \mu)$

The main result of this section is the Helson- Szegö theorem characterizing those $L^2(\mathbb{T}, \mu)$ in which the Fourier series of every function $f \in L^2(\mathbb{T}, \mu)$ converges in the norm topology. This is one of the main results of harmonic analysis on the circle group $\mathbb{T}$. It is closely related to generalized Fourier series with respect to a minimal sequence; harmonic conjugates, the Riesz projections, and weighted estimates for Hilbert singular integrals.

Definition 5.1. A sequence $(x_n)_{n\geq 1}$ in Banach Space X is called **minimal** if $x_n \notin M_n = \overline{\text{span}}\{x_k : k \neq n\}$, and is called **uniformly minimal** if $\inf\limits_{n\geq 1} \text{dist}\Big(\dfrac{x_n}{\|x_n\|}, M_n\Big) > 0$.

Lemma 5.2. *(i) A sequence $(x_n)_{n\geq 1} \subset X$ is minimal if and only if there exists $f_n \in X^*$ such that $(x_k, f_n) = \delta_{kn}$. Such a pair $((x_n)_{n\geq 1}, (f_k)_{k\geq 1})$ will be called biorthonormal and f_n, $n \geq 1$ coordinate functionals.*

(ii) $(x_n)_{n\geq 1} \subset X$ is uniformly minimal if and only if there exists a sequence $(f_n)_{n\geq 1}$ of coordinate functionals such that $\sup\limits_{n\geq 1} \|x_n\|\,\|f_n\| < \infty$.

Proof. (i) By Hahn-Banach theorem, if $x_n \notin M_n$, then there exists a sequence $f_n \in X^*$ with $\|f_n\| = 1$, $f_n(x_n) = \|x_n\|$, $\tilde{f}_n(x_n) = 1$, $\tilde{f}_n = \frac{f_n}{\|x_n\|}$.

(ii) Moreover for any subspace $E \subset X$,

$$\text{dist}(x, E) = \sup\{|f(x)| : f \in X^*,\ f|_{\mathbb{E}} \equiv 0,\ \|f\| \leq 1\}.$$

For this, if $x \in E$ then both sides are equal. So firstly we will show "$\leq$". When $x \notin E$, by Hahn- Banach theorem there exists $\tilde{f} \in X^*$ such that $\tilde{f}(x) = \text{dist}(x, E)$, and $\tilde{f}(E) = 0$ with $\|\tilde{f}\| \leq 1$. Implies

$$\text{dist}(x, E) = |\tilde{f}(x)| \leq \sup\{|f(x)|\ :\ f \in X^*,\ f|_{\mathbb{E}} \equiv 0,\ \|f\| \leq 1\}.$$

For the other inequality, let $y \in E$, then we have

$$|f(x)| = |f(x - y)| \leq \|f\|\|x - y\| \leq \|x - y\|,$$

and hence $|f(x)| \leq \inf\limits_{y\in E} \|x - y\| = \text{dist}(x, E)$. This implies

$$\sup\{|f(x)| : f \in X^*,\ f|_{\mathbb{E}} \equiv 0,\ \|f\| \leq 1\} \leq \text{dist}(x, E).$$

Thus,

$$\sup\{|f(x)| : f \in X^*, \ f|_{\mathbb{E}} \equiv 0, \ \|f\| \le 1\} = \operatorname{dist}(x, E).$$

Now, replacing f by $f/f(x)$, it follows that

$$\inf\Big\{\|f\| : f \in X^*, \ f|_{\mathbb{E}} \equiv 0, \ f(x) = 1\} = \frac{1}{\operatorname{dist}(x, E)}.$$

Apply this to $x = x_n$, $E = M_n$, and let $f_n \in X^*$ be the corresponding coordinate functionals with minimal norm. Then,

$$\operatorname{dist}\Big(\frac{x_n}{\|x_n\|}, M_n\Big) = \frac{1}{\|x_n\|}\operatorname{dist}(x_n, M_n) = \frac{1}{\|x_n\|}\frac{1}{\|f_n\|}.$$

Thus,

$$\inf_{n\ge1} \operatorname{dist}\Big(\frac{x_n}{\|x_n\|}, M_n\Big) > 0 \text{ if and only if} \sup_{n\ge1} \|x_n\| \, \|f_n\| < \infty.$$

$\square$

Definition 5.3. To a minimal sequence (x_n) we associate the (formal) Fourier series

$$x \sim \sum_{n\ge1} (x, f_n)x_n, \ x \in X.$$

The operator $x \longmapsto P_n x = (x, f_n)x_n$ is called the projection on the n^{th} Fourier component (or the co-ordinate projection with respect to the biorthogonal pair $((x_n)_{n\ge1}, (f_k)_{k\ge1})$.

Remark 5.4. We have $\|P_n\| = \|f_n\|\|x_n\|$ (because $f_n(x_n) = 1$).

Definition 5.5. A sequence (x_n) in Banach space X is called a basis of X if for all $x \in X$ there exists a unique sequence $(a_n) \subset \mathbb{C}$ such that $x = \sum_{k\ge1} a_k x_k$. Note that $a_n = a_n(x)$ A sequence x_n is called a basis sequence if it is basis in $\overline{\operatorname{span}}_X\{x_n : n \ge 1\}$.

Theorem 5.6. (*S. Banach, 1932*) *Let (x_k) be a basis of the Banach space X. Then (x_k) is uniformly minimal and $f_k(x) = a_k(x)$, $x \in X$ are the coordinate functionals.*

Definition 5.7. Let X be a Banach space and let $(x_n)_{n\in\mathbb{Z}}$ be a family in X. Then it is called **symmetric basis** if for all $x \in X$, there exists a unique $(a_k(x))_{k\in\mathbb{Z}} \subset \mathbb{C}$ such that $x = \lim_{n\to\infty} \sum_{k=-n}^{n} a_k(x)x_k$. It is called **non-symmetric** if $x = \lim_{n,m\to\infty} \sum_{k=-m}^{n} a_k(x)x_k$.

Lemma 5.8. *Let $\chi = (x_k)_{k\in\mathbb{Z}}$ and $(f_k)_{k\in\mathbb{Z}}$ be a biorthogonal pair in a Banach space X. Set $P_{m,n} = \sum_{k=-m}^{n} (.\,, f_k)x_k,\ \ m,n \in \mathbb{Z}$. Then*

(i) χ is a symmetric (respectively non-symmetric) basis if and only if $\sup_{n\geq 1} \|P_{-n,n}\| < \infty$ (respectively $\sup_{m,n} \|P_{m,n}\| < \infty$) and χ is complete.

(ii) If χ is a (at least symmetric) basis, then $(f_k)_{k\in\mathbb{Z}}$ is total, i.e. $f_k(x) = 0$ for all $k \in \mathbb{Z}$ implies $x = 0$.

(iii) For $\sigma \subset \mathbb{Z}$, define $\chi_\sigma = \overline{span}\{x_k : k \in \sigma\}$ and $\chi^\sigma = \overline{span}\{x \in X : f_k(x) = 0 \text{ for all} k \notin \sigma\}$. If χ is a basis, then for all $\sigma \subset \mathbb{Z}$, we have $\chi_\sigma = \chi^\sigma$.

Proof. (i) is followed from the Banach-Steinhaus theorem, and the fact that $\lim_{m,n} P_{m,n}x = x$ for all $x \in \mathcal{L}in\{x_k : k \in \mathbb{Z}\}$.

(ii) If $f_k(x) = 0$ for all $k \in \mathbb{Z}$, then $P_{-n,n}x = 0$ for all $n \geq 1$. Hence $x = 0$.

(iii) The inclusion $\chi_\sigma \subset \chi^\sigma$ is clear (even for minimal families). On the other hand, if $x \in X^\sigma$, then $x = \lim_{n\to\infty} P_{-n,n}x$ with $P_{-n,n}x \in X_\sigma$. Hence $x \in X_\sigma$.

□

5.1. **Skew projections.** Let L, M be two subspaces of a vector space X such that $L \cap M = \{0\}$. Define $P : L + M \to X$ by $P(x + y) = x$, then $P^2 = P$, $P|_L = id$ and $P|_M = 0$. Then P is called **skew projection** onto L parallel to M and denoted as $P := P_{L||M}$.

Lemma 5.9. *Let L, M be two subspaces of a Banach space X verifying $L \cap M = \{0\}$. Then*

(i) $P_{L||M}$ is continuous if and only if $P_{\bar{L}||\bar{M}}$ is well defined and continuous (here $\bar{L} = clos\ L$ and $\bar{M} = clos\ M$).

(ii) If L, M are closed, then $P_{L||M}$ is continuous if and only if $L + M = clos\ (L + M)$.

Proof. (i) is clear from the definition, and (ii) follows from the closed graph theorem. □

Definition 5.10. Let L, M be two subspaces of a Hilbert space H. Define angle $\alpha \in [0, \frac{\pi}{2}]$ (or minimal angle) between L and M by

$$\cos \alpha = \sup_{x \in L,\, y \in M} \frac{|\langle x, y\rangle|}{\|x\|\|y\|}.$$

NOTATION: We write $\alpha = \langle L, M\rangle$.

Remark 5.11. $L \perp M$ if and only if $\alpha = \frac{\pi}{2}$.

Lemma 5.12. *With the above notations we have*

$$\cos\langle L, M\rangle = \cos\langle \bar{L}, \bar{M}\rangle = \|P_{\bar{M}} P_{\bar{L}}\|$$

and

$$\sin\langle L, M\rangle = \sin\langle \bar{L}, \bar{M}\rangle = \|P_{L||M}\|^{-1},$$

where the symbols have obvious meaning.

Proof. Clearly, $\sup_{y \in M\setminus\{0\}} \frac{|(P_{\bar{M}}x, y)|}{\|y\|} = \|P_{\bar{M}}x\|$. Moreover, $\langle x, y\rangle = \langle P_{\bar{M}}x, y\rangle$ for $y \in M$ and hence

$$\cos\langle L, M\rangle = \sup_{0\neq x \in L,\, 0\neq y \in M} \frac{|\langle x, y\rangle|}{\|x\|\|y\|} = \sup_{0 \neq x \in L} \frac{\|P_{\bar{M}}x\|}{\|x\|}.$$

But

$$\sup_{0\neq x \in L} \frac{\|P_{\bar{M}}x\|}{\|x\|} = \sup_{0\neq x \in L} \frac{\|P_{\bar{M}}P_{\bar{L}}x\|}{\|x\|} = \sup_{0\neq x \in H} \frac{\|P_{\bar{M}}P_{\bar{L}}x\|}{\|x\|} = \|P_{\bar{M}}P_{\bar{L}}\|.$$

Next,

$$\|P_{L||M}\|^2 = \sup_{x \in L,\, y \in M} \frac{\|x\|^2}{\|x+y\|^2} = \sup_{x \in L} \frac{\|x\|^2}{\inf_{y \in M}\|x+y\|^2} = \sup_{0\neq x \in L} \frac{\|x\|^2}{\|(1-P_{\bar{M}})x\|^2}.$$

This now gives

$$\sin^2\langle L, M\rangle = 1 - \cos^2\langle L, M\rangle = 1 - \sup_{0\neq x\in L} \frac{\|P_{\bar{M}}x\|^2}{\|x\|^2} = \inf_{0\neq x \in L} \frac{\|(1-P_{\bar{M}})x\|^2}{\|x\|^2} = \frac{1}{\|P_{L||M}\|^2}.$$

□

Corollary 5.13. *The projection $P_{L||M}$ is continuous if and only if $\|P_{\bar{L}}P_{\bar{M}}\| < 1$ (and hence if and only if $\langle L, M\rangle > 0$). Moreover, $\|P_{L||M}\| = \|P_{M||L}\|$.*

5.2. **Bases of exponentials in $L^2(\mathbb{T}, \mu)$.** Now, let $X = L^2(\mathbb{T}, \mu)$, where μ is a finite Borel measure, and $x_k = e^{ikt}$, $k \in \mathbb{Z}$ (or $x_k = z^k$, $k \in \mathbb{Z}$).

Lemma 5.14. *If $(e^{ikt})_{k\in\mathbb{Z}}$ is a basis of $L^2(\mu)$ (at least in the sense of symmetric sums), then $\mu_s \equiv 0$.*

Proof. Let $\sigma_n = \{k : k \geq n\}$, let $L^2_{\sigma_n} = \overline{\text{span}}_{L^2(\mu)}\{z^k : k \geq n\}$, and let f_k be coordinate functionals associated to $(e^{ikt})_{k\in\mathbb{Z}}$, then

$$\bigcap_{n\geq 1} L^2_{\sigma_n} = \{x \in L^2(\mu) : f_k(x) = 0 \text{ for all} k \in \mathbb{Z}\} = \{0\}$$

(by Banach theorem 5.6). Clearly, $L^2_{\sigma_n}$ is an invariant subspace, and $z^n \in L^2_{\sigma_n}$ and $z^n \neq 0$ on $\mathbb{T}$. So it can be deduced (as in Corollary 2.1) that $L^2_{\sigma_n} = L^2_{\sigma_n}(\mu_a) + L^2(\mu_s)$ for all $n \in \mathbb{Z}$. But then also, $\bigcap_{n\geq 1} L^2_{\sigma_n} \supset L^2(\mu_s)$, implies $L^2(\mu_s) = 0$. □

Remark 5.15. For studying exponential basis in $L^2(\mathbb{T},\mu)$ one can restrict to measure which is absolutely continuous with respect to the Lebesgue measure m, $d\mu = wdm$, $w \in L^1_+(\mathbb{T}, m)$.

Lemma 5.16. (*Kolmogorov, 1941*) *Let $w \geq 0$, $w \in L^1_+$. Then $(z^n)_{n\in\mathbb{Z}}$ is a minimal sequence in $L^2(wdm)$ if and only if $\frac{1}{w} \in L^2(\mathbb{T})$.*

Proof. Due to biorthogonality, we have

$$\delta_{n,k} = (z^n, f_k)_{L^2(\mu)} = \int_{\mathbb{T}} z^n \bar{f}_k w dm,\ n, k \in \mathbb{Z}.$$

So we deduce that $\bar{f}_k w = \bar{z}^k$, $k \in \mathbb{Z}$, that is $f_k = \frac{z^k}{w}$, $k \in \mathbb{Z}$ (if the coordinate functional exists) Hence,

$$f_k \in L^2(wdm) \text{ if and only if } \int_{\mathbb{T}} \frac{1}{w^2} w dm < \infty.$$

□

5.3. **A fundamental reduction.** Let $\mathbb{P}, \mathbb{P}_+$ be as earlier and $\mathbb{P}_- = \text{span}\{e^{ikt} : k < 0\}$. Define the Riesz projection P_+ by

$$P_+ f = \sum_{k\geq 0} \hat{f}(k) e^{ikt},\ f \in \mathbb{P}.$$

Then

$$P_+ = P_{\mathbb{P}_+ || \mathbb{P}_-}.$$

Let also

$$P_{m,n}f = \sum_{k=m}^{n} \hat{f}(k)e^{ikt},\ f \in \mathbb{P},\ m,n \in \mathbb{Z},\ m \leq n.$$

The following result gives the principle link between the problem of bases and the norm estimation of the Riesz projection.

Lemma 5.17. *Let $w \in L^1_+$. Then the followings are equivalent.*

(i) $(z^k)_{k\in\mathbb{Z}}$ *is a nonsymmetric basis of* $L^2(wdm)$.

(ii) $\sup\limits_{n,m\in\mathbb{Z}} \|P_{m,n}\| < \infty$.

(iii) $(z^k)_{k\in\mathbb{Z}}$ *is a symmetric basis of* $L^2(wdm)$.

(iv) $\sup\limits_{n\in\mathbb{Z}} \|P_{-n,n}\| < \infty$.

(v) The Riesz projection P_+ *is continuous on* $L^2(wdm)$.

(vi) $\langle P_+, P_-\rangle > 0$ *(or* $\langle H^2_+, H^2_-\rangle > 0$, *where* $H^2_\pm$ $clos_{L^2(wdm)}\mathbb{P}_\pm$.

Proof. In view of Lemma 5.8 we get (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). It is also clear that (ii) implies (iv). Using Lemma 5.12 and Corollary 5.13 we obtain (v) $\Leftrightarrow$ (vi). Next, we verify that (iv) implies (v). Pick $f \in \mathbb{P}$, then for $n = n(f)$ sufficiently large, we get $P_+f = z^nP_{-n,n}z^{-n}f$, so $\|P_+f\| = \|P_{-n,n}z^{-n}f\| \leq \|P_{-n,n}\|\|f\|$ implies $\|P_+\| \leq \sup\limits_{n\geq 1} \|P_{-n,n}\|$. It remains to show that (v) implies (ii). Note that

$$P_{m,n}f = z^{n+1}(1 - P_+)z^{-(n+m+1)}P_+z^mf, \quad f \in \mathbb{P}.$$

But, then

$$\|P_{m,n}f\| = \|(1 - P_+)z^{-(n+m+1)}P_+z^mf\| \leq \|P_+\|\|P_+z^mf\| \leq \|P_+\|^2\|f\|$$

for all $f \in \mathbb{P}$. Since $\|1 - P_+\| = \|P_+\|$, (by Corollary 5.13) the result follows. $\square$

5.4. **Harmonic conjugates.** In order to get the desired characterization of exponential type bases in $L^2(\mu)$, we need a result of analytic type, namely, the so-called harmonic conjugation on $\mathbb{T}$(or $\mathbb{D}$).

Theorem 5.18. *Let $u \in L^2(\mathbb{T})$ be a real valued function. Then there exist a unique real valued function $v \in L^2(\mathbb{T})$ such that $\hat{v}(0) = 0$ and $u + iv \in H^2$. The mapping $u \mapsto v$ is linear and continuous with $\|v\| \leq \|u\|$.*

Proof. Let $u = \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{int} \in L^2$. Then $\bar{u} = \sum_{n\in\mathbb{Z}} \bar{\hat{u}}(n)e^{-int}$. Since u is real valued, $\bar{u} = u \Leftrightarrow \bar{\hat{u}}(n) = \hat{u}(-n)$, $n \in \mathbb{Z}$. Define

$$f = \hat{u}(0) + 2\sum_{n\geq 1} \hat{u}(n)z^n.$$

Then $f \in H^2$ and

$$\text{Re} f = \frac{1}{2}(f + \bar{f}) = \hat{u}(0) + \sum_{n\geq 1} \hat{u}(n)e^{int} + \sum_{n\geq 1} \bar{\hat{u}}(n)e^{-int} = u.$$

This means that $v = \text{Im}\, f$ will satisfy the conclusion of the theorem. Next, we show that v is unique. If $u+iv = u+iv_1 \in H^2$, then $v-v_1 \in H^2$. As $v-v_1$ is real valued $\overline{v - v_1} \in H^2$. But this is possible only if $v - v_1 = c$. Also $c = \hat{v}(0) - \hat{v}_1(0) = 0$. Finally, we have

$$v = \text{Im} f = \frac{f - \bar{f}}{2i} = \frac{1}{i}\Big(\sum_{n\geq 1} \hat{u}(n)e^{int} - \sum_{n\geq 1} \bar{\hat{u}}(n)e^{-int}\Big) = \frac{1}{i}\Big(\sum_{n>0} \hat{u}(n)e^{int} - \sum_{n<0} \hat{u}(n)e^{int}\Big).$$

The process $u \longmapsto v$ is linear and

$$\|v\|^2 = \sum_{k\neq 0} |\hat{u}(k)|^2 \leq \|u\|^2,$$

and if $\hat{u}(0) = 0$, then $\|u\| = \|v\|$. $\square$

Definition 5.19. The function v is called Harmonic conjugate of u. Let $v = \tilde{u}$. The mapping $H : L^2(\mathbb{T}) \to L^2(\mathbb{T}),\ u \longmapsto \tilde{u}$ is called the **Hilbert transform**.

5.5. Different formula for $\tilde{u}$.

(a) We can translate the above formula for $\tilde{u}$ in terms of Riesz projections

$$\tilde{u} = \frac{1}{i}(P_+u - P_-u) - \frac{1}{2}\hat{u}(0).$$

In particular, if $\hat{u}(0) = 0$, then $\tilde{u} = \frac{1}{i}(P_+u - P_-u)$. Also, we have $f = u + i\tilde{u} = 2P_+u - \hat{u}(0)$.

(b) If u verify the conditions of the theorem, then $f = u + iv \in H^2$ and $u = \text{Re}\, f$. As f extends to $\mathbb{D}$ so $\text{Re} f$ does as well. For $z \in \mathbb{D}$, $u(z) = \text{Re} f * P_z = u * P_z$. Since the

Poisson kernel verify $P_z(\zeta) = \text{Re}\left(\dfrac{\zeta+z}{\zeta-z}\right)$, we get $u(z) = \text{Re}f_1(z)$, where

$$f_1(z) = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} u(\zeta)dm(\zeta).$$

Note that $f_1 \in \text{Hol}(\mathbb{D})$ and Re $f_1 = u$, $f_1(0) = \int\limits_{\mathbb{T}} udm \in \mathbb{R}$. By uniqueness, we have $f = f_1$ and

$$\tilde{u} = \text{ Im}f = \text{ Im}f_1 = \int_{\mathbb{T}} \text{Im}\Big(\frac{\zeta+z}{\zeta-z}\Big)u(\zeta)dm(\zeta) = \int_0^{2\pi} Q_r(\tau - t)u(e^{it})\frac{dt}{2\pi}$$

where $z = re^{it}$ and

$$Q_r(t) = \text{ Im}\Big(\frac{\zeta+z}{\zeta-z}\Big) = \frac{2r\sin t}{1 - 2r\cos t + r^2}.$$

Remark 5.20. For $r \to 1$, $Q_r \sim \dfrac{\sin t}{1-\cos t} = \cot(t/2)$. In fact, one can show that

$$\tilde{u}(\tau) = (u * \cot(./2))(\tau) = \int_0^{2\pi} u(\tau - t)\cot\big(t/2\big)\frac{dt}{2\pi}$$

in the sense of Cauchy principle valued integral.

5.6. **The Helson-Szegö theorem.**

Theorem 5.21. *Let μ be a finite Borel measure on $\mathbb{T}$. Then the followings are equivalent.*

- *(i) The family $(z^n)_{n\in\mathbb{Z}}$ is a (symmetric or nonsymmetric) basis of $L^2(\mu)$.*
- *(ii) The Riesz projection P_+ is bounded on $L^2(\mu)$.*
- *(iii) The angle satisfies $\sin\langle P_+, P_-\rangle > 0$.*
- *(iv) $d\mu = |h|^2dm$, where $h \in H^2$ is an outer function such that dist$\Big(\dfrac{\bar{h}}{h}, H^\infty\Big) < 1$.*
- *(v) $d\mu = wdm$, where $w = e^{u+\tilde{v}}$ and u, v are real valued bounded functions and $\|v\|_\infty < \frac{\pi}{2}$ (condition (HS)).*

The proof of the theorem will be given in several steps based on the following lemmas.

Lemma 5.22. *The mapping $j : H^2 \times H^2 \to H^1$, $(\phi,\psi) \longmapsto \phi\psi$ is continuous and symmetric. Moreover, $j(B^2 \times B^2) = B^1$, where B^p is the unit ball in H^p.*

Proof. The continuity follows from the Cauchy Schwarz inequality $\|\phi\psi\|_1 \le \|\phi\|_2\|\psi\|_2$. For surjectively, let $f \in H^1$, then $f = \lambda BS[f]$. Write $\phi = \lambda BS[f]^{\frac{1}{2}}$ and $\psi = [f]^{\frac{1}{2}}$ then $\phi\psi \in H^2$. □

Lemma 5.23. *Let E be a subspace of the Banach space X, and $\Phi \in X^*$. Then*

$$\|\Phi|_E\| = \inf\{\|\Psi\|_{X^*} : \Psi = \Phi \ onE\} = \inf\{\|\Phi + \alpha\|_{X^*} : \alpha \in X^* \ and\alpha|_E = 0\}$$

Proof. The inequality "$\leq$" is clear. For "$\geq$" apply Hahn-Banach theorem. Let $\Psi' = \Phi|_E$. Then

$$\|\Psi\|_{X^*} = \sup_{x\in X} |\Psi(x)| \geq \|\Psi'\|_{X^*} = \sup_{x\in X} |\Psi'(x)| = \|\Phi|_E\|.$$

By Hahn-Banach theorem, there exists $\Psi' \in X^*$ such that $\|\Phi|_E\| = \|\Psi'\|_{X^*}$, and hence the result follows. □

Lemma 5.24. *Let $f \in H^1$ and suppose that $f(\mathbb{T}) \subset A \subset \mathbb{C}$. Then $f(\mathbb{D}) \subset conv(A)$ (the closed convex hull of A).*

Proof. Observe that for $z \in \mathbb{D}$, we have $f(z) = P_z * f = \int\limits_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} f(\zeta) d\zeta \in \text{conv}(A)$. □

Lemma 5.25. ($V.$ *Smirnov, A. Kolmogorov*) *Let $v \in L^\infty(\mathbb{T})$ be a real valued function then $e^{\lambda\tilde{v}} \in L^1(\mathbb{T})$ if $\lambda\|v\|_\infty < \frac{\pi}{2}$.*

Proof. It is sufficient to show that $\|u\|_\infty < \frac{\pi}{2}$ implies $e^{\tilde{u}} \in L^1$. Set $f = e^{-i(u+i\tilde{u})}$, which is well defined in $\mathbb{D}$, since $u + i\tilde{u} \in H^2$. Clearly $|f| = e^{\tilde{u}}$ and $|\arg f| = |u| < \dfrac{\pi(1-\epsilon)}{2}$ for some $\epsilon > 0$ (on $\mathbb{T}$ and hence on $\mathbb{D}$ in view of Lemma 5.24). The same reasoning as in (Theorem 4.8) now gives $f \in H^1$ and hence $|f| = e^{\tilde{u}} \in L^1(\mathbb{T})$. □

Proof. **Implication (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) of Helson-Szegö theorem.**

Recall that we may restrict to $d\mu = wdm$, $w \in L^1_+(\mathbb{T})$. By Lemma 5.17 we get the equivalence of (i),(ii) and (iii).

Next we show (i) and (ii) are equivalent to (iv). Note that if the sequence $(z^n)_{n\in\mathbb{Z}}$ is a basis, then we can see from Banach's (Theorem 5.6) and Kolmogorov's (Lemma 5.16) that $\frac{1}{w} \in L^1$ and hence $\log w \in L^1$ (this can be justified without using Banach theorem as $\bar{z} \notin H^2(\mu)$ we get $\log w \in L^1$). In view of the later observation, we suppose that there exists an outer function $h \in H^2$ such that $|h|^2 = w$. Thus,

$$(f,g)_{L^2(\mu)} = \int_{\mathbb{T}} f\bar{g}wdm = \int_{\mathbb{T}} fh\bar{g}h\frac{\bar{h}h}{h^2}dm = \int_{\mathbb{T}} (fh)(\bar{g}h)\frac{\bar{h}}{h}dm = \int_{\mathbb{T}} FG\frac{\bar{h}}{h}dm$$

for all $f \in \mathbb{P}_+$ and $g \in \mathbb{P}_-$ and therefore,

$$\|f\|^2_{L^2(\mu)} = \int |fh|^2 dm = \|F\|^2_{L^2(\mathbb{T})}, \;\; \|g\|^2_{L^2(\mu)} = \|G\|^2_{L^2(\mathbb{T})}.$$

Clearly $F = fh \in H^2$, since $\bar{g} \in \mathbb{P}^0_+$, we get $G \in H^2_0$. By definition of outer function, it follows that $\overline{\text{span}}\{F = fh : f \in \mathbb{P}_+\} = H^2$, and also $A := \{F = fh : f \in \mathbb{P}_+, \|F\| \leq 1\}$ is dense in the unit ball B^2 of H^2. For the same reason, we see that $B := \{G = \bar{g}h : g \in \mathbb{P}_-, \|G\| \leq 1\}$ is dense in $B^2 \cap H^2_0$. We deduce that

$$\begin{aligned}\cos\langle \mathbb{P}_+, \mathbb{P}_-\rangle_{L^2(\mu)} &= \sup\{|(f,g)| : f \in \mathbb{P}_+, g \in \mathbb{P}_- \,\|f\|^2_L(\mu) \leq 1, \, \|g\|^2_L(\mu) \leq 1\} \\ &= \sup\Big\{\Big|\int_{\mathbb{T}} FG\frac{\bar{h}}{h}dm\Big| : F \in A, \, G \in B\Big\}.\end{aligned}$$

Set $\Phi(k) = \int\limits_{\mathbb{T}} k(\frac{\bar{h}}{h})dm$, $k \in L^2(\mathbb{T})$. As $\bar{h}/h \in L^\infty(\mathbb{T})$, we get $\Phi \in (L^1(\mathbb{T}))^*$. By (Lemma 5.22), we see that the angle $\langle \mathbb{P}_+, \mathbb{P}_-\rangle = \|\Phi|_{H^1_0}\|$, and by means of (Lemma 5.23), we can express it in terms of h:

$$\cos\langle \mathbb{P}_+, \mathbb{P}_-\rangle_{L^2(\mu)} = \|\Phi|_{H^1_0}\| = \, \text{dist}_{L^\infty(\mathbb{T})}\Big(\frac{\bar{h}}{h}, (H^1_0)^\perp\Big) = \, \text{dist}_{L^\infty(\mathbb{T})}\Big(\frac{\bar{h}}{h}, H^\infty\Big).$$

The last equality is the consequence of the relation

$$(H^1_0)^\perp = \{g \in L^\infty : \int_{\mathbb{T}} gf dm = 0 \text{ for all} f \in H^1_0\} = H^\infty.$$

Now, we conclude that $\cos\langle \mathbb{P}_+, \mathbb{P}_-\rangle < 1$ if and only if $\log w \in L^1$, $w = |h|^2$ for an outer function $h \in H^2$ satisfying $\text{dist}_{L^\infty(\mathbb{T})}(\frac{\bar{h}}{h}, H^\infty) < 1$, that is (i) and (ii) are equivalent to (iv).

Proof of implication (iv) $\implies$ (v):

Suppose $\text{dist}_{L^\infty(\mathbb{T})}(\frac{\bar{h}}{h}, H^\infty) < 1$, where h is a outer and $|h|^2 = w$. Then there exists $g \in H^\infty$ such that $\|\frac{\bar{h}}{h} - g\|_\infty < 1$. That is, for $\epsilon > 0$, we have $|\frac{\bar{h}}{h} - g| < 1 - \epsilon$ a.e. on $\mathbb{T}$, and hence $||h|^2 - gh^2| < (1-\epsilon)|h|^2$ a.e. on $\mathbb{T}$. Setting $a = |h(\xi)|^2 > 0$, for $\xi \in \mathbb{T}$, we see that $|a - gh^2| < (1-\epsilon)a$.

Geometrically, it means that if $\alpha \in (0, \frac{\pi}{2})$ is such that $\sin\alpha = 1 - \epsilon$, and $A = \{z : |\arg z| < \alpha\}$, then we get $gh^2(\mathbb{T}) \subset A$ (cf. Figure 1).

From (Lemma 5.24) we get $gh^2(\mathbb{D}) \subset A$, so $\log gh^2$ is analytic in $\mathbb{D}$. We set $v = -\text{Im} \log gh^2 = -\arg gh^2$ and get $|v| = \text{Re} \log gh^2 + c = \log|gh|^2 + c$, where c has to be chosen such that $\tilde{v}(0) = 0$. We obtain $\log gh^2 = \tilde{v} - iv - c$ and $gh^2 = e^{\tilde{v} - iv - c}$ on $\mathbb{T}$, we have $|\frac{\bar{h}}{h} - g| < 1-\epsilon$, which implies that $|1-|g|| < 1-\epsilon$, hence $\epsilon \le |g| \le 2-\epsilon$. Finally, $|h|^2 = \dfrac{e^{\tilde{v}-c}}{|g|} = e^{u+\tilde{v}}$, where $u = -\log|g| - c \in L^\infty(\mathbb{T})$ and $\|v\|_\infty < \frac{\pi}{2}$.

Proof of implication (v) implies (iv):

Let $wdm = e^{u+\tilde{v}}dm$, where $u, v \in L^\infty(\mathbb{T})$ are real valued and $\|v\|_\infty < \frac{\pi}{2}$. Clearly $\log w = u+\tilde{v} \in L^1$ and by (lemma 5.25) we have $w \in L^1(\mathbb{T})$. Hence there exists an outer function $h \in H^2$ such that $|h|^2 = w$. Thus $\log|h|^2 = u+\tilde{v}$ and $\log h^2 = u+\tilde{v}+i(u+\tilde{v})^\sim = u+\tilde{v}+i(\tilde{u}-v+c)$ for some constant $c \in \mathbb{R}$. Setting $g = e^{-(u+i\tilde{u})-ic}$ we obtain, in view of $|g| = e^{-u}$, a bounded holomorphic function $g \in H^\infty$. Moreover,

$$\frac{h}{\bar{h}}g = \frac{h^2}{|h|^2}g = \exp(i(\tilde{u}-v+c) - u - i\tilde{u} - ic) = \exp(-u-iv),$$

where $\|v\|_\infty < \frac{\pi}{2}$. This gives the following estimates on $\mathbb{T}$.

$$e^{-\|u\|_\infty} \le \left|\frac{h}{\bar{h}}g\right| \le e^{\|u\|_\infty}, \quad \left|\arg(\frac{h}{\bar{h}})g\right| = |v| < \pi\frac{(1-\epsilon)}{2}.$$

(cf. Figure 2). The value of $(\frac{h}{\bar{h}})g$ thus belongs to

$$\mathcal{D} := \left\{z \in \mathbb{C} : \, e^{-\|u\|_\infty} \le |z| \le e^{\|u\|_\infty}, \quad |\arg z| < \pi\frac{(1-\epsilon)}{2}\right\}.$$

For λ sufficiently big and some $\delta > 0$ we have $B(\lambda, (1-\delta)\lambda) \supset \operatorname{clos}\mathcal{D}$ or $\lambda^{-1}B(\lambda,(1-\delta)\lambda) = B(1, 1-\delta) \supset \lambda^{-1} \operatorname{clos}\mathcal{D}$. Then $\lambda^{-1}\frac{h}{\bar{h}}g \in B(1,1-\delta)$ a.e. on $\mathbb{T}$. In other words, $|\lambda^{-1}(\frac{h}{\bar{h}})g - 1| < 1-\delta$ a.e. on $\mathbb{T}$, and $|\lambda^{-1}g - (\frac{\bar{h}}{h})| < 1-\delta$ a.e. $\mathbb{T}$. As $g \in H^\infty$, this gives $\operatorname{dist}_{L^\infty(\mathbb{T})}(\frac{\bar{h}}{h}, H^\infty) < 1$. $\square$

5.7. **An example.** Let $\omega(e^{it}) = |t|^\alpha$, $t \in (-\pi, \pi)$, $\alpha \in \mathbb{R}$. Then for $\alpha \ge 1$ we have $1/\omega \notin L^1(\mathbb{T})$ and $(e^{int})_{n\in\mathbb{Z}}$ cannot be uniformly minimal in view of Lemma 5.16. For $\alpha \le -1$, $\omega \notin L^1$. Thus, the only interesting case is $\alpha < 1$.

First note that if the quotient ω_1/ω_2 and ω_2/ω_2 are bounded, then the sequence $(e^{int})_{n\in\mathbb{Z}}$ is a basis of $L^2(\omega_1)$ if and only if it is one of $L^2(\omega_2)$. Indeed, the identity map $f \longmapsto f$ is an isomorphism from $L^2(\omega_1)$ to $L^2(\omega_2)$.

Next, let $\omega_1 = \omega$ and $\omega_2 = (1-e^{it})^\alpha$. Then

$$\log\omega_2 = \log|1-e^{it}|^\alpha = \alpha \mathrm{Re}\arg(1-e^{it}) := u.$$

Necessarily, we get

$$
\begin{aligned}
\tilde{u}(t) &= \alpha \arg(1 - e^{it}) = \alpha \arg(e^{it/2}(e^{-it/2} - e^{it/2}) \\
&= \alpha \arg(e^{it/2}(-2i \sin t/2). \\
&= \begin{cases} \alpha(t/2 - \pi/2), & \text{if} t > 0; \\ \alpha(\pi/2 - t/2), & \text{if} t < 0. \end{cases}
\end{aligned}
$$

6. Transfer to the half-plane

In this section, we give an outline of the Hardy-space theory in the half-plane and on the line. We restrict ourselves to the key results only: an isometric correspondence between Hardy-space in the disc and in the half-plane, the canonical factorization, the Fourier transform representation (Paley-Wiener theorem), and invariant subspaces.

6.1. **A unitary mapping from $L^p(\mathbb{T})$ to $L^p(\mathbb{R})$.** Let $\omega : \mathbb{D} \to \mathbb{C}$, $\omega(z) = i\frac{1+z}{1-z}$, be the usual conformal mapping of the disc $\mathbb{D}$ to the upper half-plane $\mathbb{C}_+ = \{\xi \in \mathbb{C} : \operatorname{Im}\xi > 0\}$.

The restriction to the boundary $\omega|_{\mathbb{T}}$ is a one to one correspondence between $\mathbb{T} \setminus \{1\}$ and $\mathbb{R}$. The inverse ω^{-1}, $\omega^{-1}(x) = \dfrac{x-i}{x+i}$ has Jacobian $|J(x)| = \dfrac{2}{1+x^2}$, $x \in \mathbb{R}$. Hence the mapping $U = U_p$,

$$U_p f(x) = \left(\frac{1}{\pi(x+i)}\right)^{1/p} f(\omega^{-1}(x)),\ x \in \mathbb{R}$$

is an isomorphic isomorphism (unitary for $p = 2$) of the space $L^p(\mathbb{T})$ onto $L^p(\mathbb{R})$.

First, we give three descriptions of the image under U of the Hardy-space $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$, then pass to arbitrary p, $1 \le p \le \infty$. Clearly, $U_p H^p(\mathbb{T})$ is a closed subspace of $L^p(\mathbb{R})$.

6.2. **Cauchy kernel and Fourier transform.** The first description of $U_2 H^2(\mathbb{T})$ is straightforward.

Lemma 6.1.

$$U_2 H^2(\mathbb{T}) = \overline{span}_{L^2(\mathbb{R})}\left\{\frac{1}{x-\bar{\mu}} :\ Im\,\mu > 0\right\}.$$

Proof. Since

$$H^2(\mathbb{T}) = \overline{\operatorname{span}}_{L^2(\mathbb{T})}\left\{\frac{1}{1-\bar{\lambda}z} : |\lambda| < 1\right\},$$

and U_2 is an isometry, we have

$$H^2(\mathbb{T}) = \overline{\operatorname{span}}_{L^2(\mathbb{T})}\left\{U_2(1-\bar{\lambda}z)^{-1} = \frac{C_\lambda}{x-\overline{\omega(\lambda)}} : \lambda \in \mathbb{D}\right\}.$$

Clearly, $\mu = \omega(\lambda)$ runs over the entire upper half-plane $\mathbb{C}_+$. $\square$

Now, we recall that Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$,

$$\mathcal{F}(f)(z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ixz}dx,$$
$$\mathcal{F}^{-1}(f)(z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{ixz}dx$$

are unitary mapping of $L^2(\mathbb{R})$ onto itself.

Lemma 6.2. *$U_2H^2 = \mathcal{F}^{-1}L^2(\mathbb{R})$, where $L^2(\mathbb{R}_+) = \{f \in L^2(\mathbb{R}) : f = 0$ on $(-\infty, 0)\}$.*

Proof. Compute the inverse Fourier transform of the function $\chi_{\mathbb{R}_+}e^{i\lambda x} \in L^2(\mathbb{R}_+)$, where $\mathrm{Im}\lambda > 0$:

$$\mathcal{F}^{-1}(\chi_{\mathbb{R}_+}e^{i\lambda x}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \chi_{\mathbb{R}_+}e^{i\lambda x}e^{ixz}dx = \frac{1}{\sqrt{2\pi}}\frac{1}{i(z+\lambda)}\left[ix(z+\lambda)\right]_{x=0}^{\infty} = \frac{i}{\sqrt{2\pi}}\frac{1}{z-(-\lambda)},$$

where $-\lambda = \mu$ runs, again, over the entire half-plane $\mathbb{C}_+$. Since $\mathcal{F}^{-1}$ is an isometry, Lemma 6.2 reduces to the proof of the following identity:

$$L^2(\mathbb{R}_+) = \overline{\mathrm{span}}\{\chi_{\mathbb{R}_+}e^{i\lambda x}\ \mathrm{Im}\lambda > 0\}.$$

The equality follows from the injectivity (classical Fourier uniqueness theorem) of the Fourier transform $\mathcal{F}$. Namely, let $f \in L^2(\mathbb{R}_+)$ and suppose that $f \perp \chi_{\mathbb{R}_+}e^{i\lambda x}$, for all λ with $\mathrm{Im}\,\lambda > 0$. Taking $\lambda = i + y$, $y \in \mathbb{R}$, we get $\mathcal{F}(f\,\chi_{\mathbb{R}_+}e^{-x}(y) = 0$ for all $y \in \mathbb{R}$. Hence $f\,\chi_{\mathbb{R}_+}e^{-x} = 0$ a.e. on $\mathbb{R}$ and so $f = 0$. □

6.3. **The Hardy space $H^p_+ = H^p(\mathbb{C}+)$.** Here we see from real line $\mathbb{R}$ to the half-plane $\mathbb{C}_+$. We identify the subspace $U_pH^p \subset L^2(\mathbb{R})$ with the space of boundary values of a certain holomorphic space in the half-plane C_+. Note that $\omega^{-1}(z) = \frac{z-i}{z+i}$ is a conformal mapping from $\mathbb{C}_+$ to $\mathbb{D}$. Hence the same formula as above,

$$U_pf(z) = \left(\frac{1}{\pi(z+i)}\right)^{1/p} f(\omega^{-1}(z)),\ \mathrm{Im}\,z > 0$$

defines a holomorphic function in $\mathbb{C}_+$ for all $f \in H^p(\mathbb{C}_+)$. Moreover, ω^{-1} is still conformal at the boundary points $r \in \mathbb{R}$ and transfers a Stolz angle in $\mathbb{C}_+$, $\{x+iy : |x-r| < cy\}$, into a Stolz angle in $\mathbb{D}$. Now, Fatou's theorem implies that the functions U_pf, $f \in H^p(\mathbb{D})$, have non-tangential boundary limits $(U_p(f))_{\mathbb{R}}$ a.e. on $\mathbb{R}$, $U_p(f_{\mathbb{T}}) = (U_pf)_{\mathbb{R}}$. Hence in order to get another characterization of $U_pH^p(\mathbb{T})$, it remains to describe $U_pH^p(\mathbb{D})$ in intrinsic terms as a subset of $\mathrm{Hol}(\mathbb{C}_+)$. This is done in the next theorem. But, first we define Hardy classes on $\mathbb{C}_+$.

Definition 6.3. Hardy space $H^p_+ = H^p(\mathbb{C}_+)$, $0 < p \leq \infty$, is the class of functions $g \in \mathrm{Hol}(\mathbb{C}_+)$ such that

$$\|g\|_{H^p_+} = \sup_{y>0} \Big(\int_{\mathbb{R}} |g(x+iy)|^p dx \Big)^{\frac{1}{p}} < \infty,$$

with the usual modification for $p = \infty$. In order to compare $H^p(\mathbb{C}_+)$ with $U_p H^p(\mathbb{D})$, we need the following simple result.

Lemma 6.4. *(i) Let γ be an arbitrary circle in $\overline{\mathbb{D}}$. Then*

$$\int_\gamma |f(z)|^p |dz| \leq 2 \int_{\mathbb{T}} |f(z)|^p |dz|$$

for all $f \in H^p(\mathbb{D})$, $1 \leq p < \infty$, here $|dz|$ stands for the arc length measure.

(ii) Let $g \in H^p(\mathbb{C}_+)$, $1 \leq p < \infty$ and $z \in \mathbb{C}_+$, then

$$|g(z)| \leq \Big(\frac{2}{\pi Im\, z} \Big)^{\frac{1}{p}} \|g\|_{H^p_+}.$$

Proof. (i) First let $p = 1$. For $x \in L^2(\mu)$, denote by u_* be the harmonic extension of u in the unit disc,

$$u_*(z) = \int_{\mathbb{T}} u(\zeta) \frac{1-|z|^2}{|\zeta - z|^2} dm(\zeta),\ z \in \mathbb{D}.$$

We show that $u \longmapsto u_*|_\gamma$ is a bounded operator from $L^1(\pi)$ to $L^1(\gamma)$ of norm at most 4π. Indeed,

$$\begin{aligned} \int_\gamma |u_*(z)||dz| &\leq \int_\gamma |u(\zeta)| \frac{1-|z|^2}{|\zeta - z|^2} dm(\zeta)|dz| \\ &= \int_{\mathbb{T}} |u(\zeta)| \Big(\int_\gamma \frac{1-|z|^2}{|\zeta - z|^2} |dz| \Big) dm(\zeta) \\ &= 2\pi r \int_{\mathbb{T}} |u(\zeta)| \frac{1-|c|^2}{|\zeta - c|^2} dm(\zeta), \end{aligned}$$

where $\gamma = \gamma(c, r)$. In the last inequality, we have used the MVT for harmonic functions applied to the Poisson kernel $P_z(\zeta) = \mathrm{Re}\left(\frac{z+\zeta}{z-\zeta}\right)$. Since $2\pi dm(z) = |dz|$ on $\mathbb{T}$, $r \leq 1 - |c|$ and $\frac{1-|c|^2}{|\zeta-c|^2} \leq \frac{1+|c|}{1-|c|} \leq \frac{2}{1-|c|}$, we get the desired inequality. For an arbitrary p, $1 \leq p < \infty$, we have $|u_*|^p \leq (|u|^p)_*$, from Holder's inequality, and the result follows.

(ii) Using the MVT in the disc, $D = \{x + iy : |\lambda - (x + iy)| < \ \mathrm{Im}\lambda\}$, Holder's inequality, and what is sometimes called the "rolling a disk" trick:

$$\begin{aligned} |g(\lambda)| &= \frac{1}{\pi(\mathrm{Im}\lambda)^2}\int_D |g\,dxdy| \le \Big(\frac{1}{(\pi\mathrm{Im}\lambda)^2}\Big)^{1-\frac{1}{p}}\Big(\int_D |g|^p dxdy\Big)^{\frac{1}{p}} \\ &\le \Big(\frac{1}{\pi(\mathrm{Im}\lambda)^2}\Big)^{\frac{1}{p}}\Big(\int_0^{2\mathrm{Im}\lambda} dy\int_{\mathbb{R}} |g(x+iy)|^p dy\Big)^{\frac{1}{p}} \\ &\le \Big(\frac{2}{(\pi\mathrm{Im}\lambda)}\Big)^{\frac{1}{p}}\|g\|_{H^p_+}. \end{aligned}$$

□

The following theorem is one of the main result of this section.

Theorem 6.5. *Let $1 \le p \le \infty$. Then $U_p H^p(\mathbb{D}) = H^p(\mathbb{C}_+)$.*

Proof. If $g \in$ Hol $(\mathbb{C}_+)$, $y > 0$, and $Uf = g$, then

$$\int_{\mathbb{R}} |g(x+iy)|^p dx = \frac{1}{2\pi}\int_{C_y} |f(z)|^2 |dz|,$$

Where C_y is the circle in $\mathbb{D}$ having the interval $[y - 1/y + 1, 1]$ as diameter and being tangent to the unit circle $\mathbb{T}$ at the point 1. So it remains to verify that

$$\sup_{0<r<1}\int_{\mathbb{T}} |f(r\xi)|^p |d\xi| < \infty \Leftrightarrow \sup_{y>0}\int_{C_y} |f|^p |dz| < \infty,$$

for every $f \in \mathrm{Hol}\,(\mathbb{D})$.

The implication $\Longrightarrow$ is a straightforward implication of Lemma 6.4 (i).

To prove the converse, let $g \in H^p_+$. By Lemma 6.4(ii), g is bounded on every half-plane Im $\ge y > 0$. Hence $g \circ w$ is bounded on the disc int(C_y). Since the function $(1 - z)^{-1}$ is outer on the int(C_y) and $f = \pi\Big(\big(\frac{2i}{1-z}\big)^2\Big)^{\frac{1}{p}}(g\,o\,w) \in L^p(C_y)$, we get $f \in H^p(C_y)$ by the integral maximum principle 3.26(iv). (We use the previous theory for the following classes $H^p(D)$ over disc $\mathbb{D} =$ int (C_y), instead of the unit disc $\mathbb{D}$; the corresponding modifications, including the very definition of $H^p(\mathbb{D})$, do not cause any difficulties and can be obtained by a linear change of variable). Now, applying Lemma 6.4(i) to the circle $\gamma(r) = \{z \in \mathbb{C} : |z| = r\} \subset$ int(C_y), we get

$$\int_{\gamma(r)} |f(z)|^p |dz| \le 2\sup_{y>0}\int_{C_y} |f(z)|^p |dz|.$$

In fact, the Poisson representation (Corollary 6.7) implies that for $g \in H^p_+$, the norms

$$\Big(\int_{\mathbb{R}} |g(x+iy)|^p dx\Big)^{\frac{1}{p}}$$

are monotonically increasing in $y > 0$ and tend to $\|g|_{\mathbb{R}}\|_{L^p}$ as $y \to 0$ (to see this, use approximate identity properties of the Poisson kernel). This shows that $\|g|_{\mathbb{R}}\|_{L^p} = \|g\|_{H^p_+}$.

□

Theorem 6.6. (*R. Paley and N. Wiener, 1934*)

$$H^p(\mathbb{C}_+) = \mathcal{F}^{-1} L^2(\mathbb{R}_+)$$

Proof. This is immediate from Lemma 6.2 and Theorem 6.5. □

6.4. **Canonical factorization and other properties:** The following properties are straightforward consequences of the change of variables from Section 6.1, Theorem 6.5, and the corresponding facts from H^p theory in the disc $\mathbb{D}$.

Corollary 6.7. (*Poisson formula*) *If $f \in H^p(\mathbb{C}_+)$, $1 \leq p \leq \infty$, then*

$$f(x+iy) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2} f(t)dt, \; y > 0.$$

Corollary 6.8. (*Boundary uniqueness theorem*) *If $f \in H^p(\mathbb{C}_+)$, $1 \leq p \leq \infty$ and $f \neq 0$, then*

$$\int_{\mathbb{R}} \frac{|\log|f(x)||}{1+x^2} dx < \infty.$$

Corollary 6.9. (*Blaschke condition and Blaschke product*) *If $f \in H^p(\mathbb{C}_+)$, $1 \leq p \leq \infty$, and $f \neq 0$, then*

$$\sum \frac{Im\lambda_n}{1+|\lambda_n|^2} < \infty,$$

where λ_n are the zero of f in $\mathbb{C}_+$ (counting multiplicities). The corresponding Blaschke product (having similar properties as in $\mathbb{D}$) is

$$B(z) = \prod_n \epsilon_n \frac{z-\lambda_n}{z-\bar{\lambda}_n}, \; z \in \mathbb{C}_+,$$

where $\epsilon_n = \frac{|\lambda_n^2+1|}{\lambda_n^2+1}$ (by definition, $\epsilon_n = 1$ for $\lambda_n = i$).

Theorem 6.10. *Each function $f \in H^p(\mathbb{C}_+)$; $1 \leq p \leq \infty$, has a unique factorization of the form $f = \lambda BV[f]$, where $\lambda \in \mathbb{T}$, B is the Blaschke product constructed from the zeroes of f, V is a singular inner function (an H^∞ function having no zeroes in $\mathbb{C}_+$ and with unimodular boundary values on $\mathbb{R}$) of the form*

$$V(z) = e^{iaz} V_v(z) = e^{iaz} \exp\left(i \int_{\mathbb{R}} \frac{1+tz}{t-z} dv(t) \right),$$

where $a \geq 0$, and v is a finite positive singular measure on $\mathbb{R}$, $[f]$ is the Schwarz-Herglotz outer factor of the form

$$[f](z) = \exp\left(\frac{1}{\pi i} \int_{\mathbb{R}} \frac{1+tz}{t-z} \log|f(t)| \frac{dt}{1+t^2} \right), \ z \in \mathbb{C}_+$$

Proof. Just a change variable in the Riesz Smirnov Theorem 3.25. The only new detail concerns the factor e^{iaz}. Namely, changing variables in the integral for the singular inner factor S of Theorem 3.25. We have to take care of the exceptional point $1 \in \mathbb{T}$, since it may carry a positive mass, say $a > 0$. To do this, it is natural to extend the mapping w^{-1} to a bijection of $\mathbb{R} \cup \{\infty\}$ to $\mathbb{T}$ by simple setting $w^{-1}(\infty) = 1$. Now, the point mass $a\delta$, of the measure μ at 1 turns into the point mass $a\delta_\infty$ of v at ∞, and replacing the measure μ on $\mathbb{T}$ by its preimage $v = w^{-1}\mu$ on $\mathbb{R} \cup \{\infty\}$, we get e^{iaz} as a part of the integral for $V = S \circ w^{-1}$. We prefer to separate the point mass at infinity and write $V = e^{iaz} V_v$. $\square$

Remark 6.11. It is clear from the previous computations that other facts of the Hardy Nevanlinna theory of Sections 3 and 4 in the disc can be transferred to the half-plane. In particular, the properties of the inner outer factorization from subsections 4.2-4.4 still hold with corresponding modifications caused by the change of variables. For instance, a function $f \in H^p(\mathbb{C}_+)$ having an analytic continuation across a point $x \in \mathbb{R}$ has singular representing measure zero in a neighborhood of this point. To find the point mass of the singular measure, the logarithmic residues of Section 4 (to be added) can be redefined and computed and so on and so on. In particular, the point mass at ∞ is $a = -\lim_{y\to\infty} \frac{1}{y} \log|f(iy)|$.

6.5. **Invariant subspaces.** Here we consider translation invariant subspaces of $L^2(\mathbb{R})$ and their Fourier dual objects - character invariant subspaces.

6.6. **Duality between translation and multiplication by characters.** Define the translation operator τ_s by

$$(\tau_s f)(x) = f(x-s),\ x \in \mathbb{R},\ \text{for } s \in \mathbb{R}.$$

This is a group of unitary operators on $L^2(\mathbb{R})$. A subspace $E \subset L^2(\mathbb{R})$ (closed, as always) is said to be (translation) 2-invariant and if $\tau_s E \subset E$ for all $s \in \mathbb{R}$, and (translation) 1-invariant if $\tau_s E \subset E$ for all $s \geq 0$ but not for (all) $s < 0$. The Fourier image of the translation operator τ_s is the multiplication operator by the corresponding character e^{isx} of the group $\mathbb{R}$:

$$\tau_s(\mathcal{F}f) = \mathcal{F}(e^{is}f),\ \text{for all } f \in L^2(\mathbb{R}).$$

Without any risk of confusion, we write e^{isx} both for the function $x \longmapsto e^{isx}$ and for the multiplication operator by this function, $f \longmapsto e^{isx}f$. Hence, we have

$$\tau_s = \mathcal{F}e^{isx}\mathcal{F}^{-1},$$

that is, the groups $(\tau_s)_{s\in\mathbb{R}}$ and $(e^{isx})_{s\in\mathbb{R}}$ are unitarily equivalent (conjugate) via the Fourier transform.

We use the same terminology as above for e^{isx} -invariant subspaces. A subspace $E \subset L^2(\mathbb{R})$ is (character) 2-invariant if $e^{isx}E \subset E$ for all $s \in \mathbb{R}$, and (character) 1-invariant if $e^{isx}E \subset E$ for $s \geq 0$ but for (all) $s < 0$. Hence, E is an 1- or 2- character invariant if and only if its Fourier image $\mathcal{F}E$ is a 1- or 2- translation invariant subspace.

Clearly, the Hardy space $H^2(\mathbb{C}_+)$ is a character 1-invariant subspace, and $\mathcal{F}H^2(\mathbb{C}_+) = L^2(\mathbb{R}_+)$ is translation 1-invariant.

Below, we will derive analogue of the Wiener theorem 1.4 and Beurling Helson theorem 1.5 for character invariant subspaces. First, we prepare the transfer of these results to $L^2(\mathbb{R})$ by means of the operator U_2.

Lemma 6.12. *Let* $u_s = \exp\left(s\frac{z+1}{z-1}\right)$ $s \in \mathbb{R}$, *and let* E *be a (closed) subspace of* $L^2(\mathbb{R})$. *The* E *is a 2-invariant subspace (with respect to the shift operator* $f \longmapsto zf$*) if and only if* $u_s E \subset E$, *for all* $s \in \mathbb{R}$, *and* E *is 1-invariant subspace if and only if* $u_s E \subset E$, *for all* $s \geq 0$, *but not for (all)* $s < 0$.

Proof. If $b \in H^\infty$, and E is a z-invariant subspace of $L^2(\mathbb{T})$, then $bE \subset E$. Indeed, by DCT, we have

$$\lim_{r\to 1} \|bf - b_r f\|_2 = 0, \text{ for all } f \in E,$$

where $b_r(z) = b(rz)$.

On the other hand, $z^n f \in E$, for $n \geq 0$ and therefore, $b_r f \in E$, since Taylor series of b_r is absolutely convergent on $\mathbb{T}$. Hence $bf \in E$. The same holds true for $\bar{b} \in H^\infty$ and $\bar{z}-$invariant subspace E. These prove the "only if" part of the lemma.

By analogous reasoning, to prove the converse, it suffices to show that the function z is the bounded pointwise limit of functions $\phi_s = \dfrac{u_s - (1-s)}{u_s - (1+s)}$ as $s \to 0_+$. We have $\mathrm{Re}(1 - u_s(\zeta)) \geq 0$, and hence $|\phi_s(\zeta)| \leq 1$, for $\zeta \in \mathbb{T}$. On the other hand, using the standard formula

$e^{sw} = 1 + sw + o(s)$ as $s \to 0_+$, we easily get $\lim_{s\to 0} \phi_s(\zeta) = \zeta$ for $\zeta \in \mathbb{T} \setminus \{1\}$. □

Theorem 6.13. (*P. Lax, 1959*) *Let* E *be a subspace of* $L^2(\mathbb{R})$.

(i) E *is a (character) 2-invariant subspace if and only if* $E = \chi_\Sigma L^2(\mathbb{R})$ *for a measurable subset* $\Sigma \subset \mathbb{R}$.

(ii) E *is a (character) 1-invariant subspace if and only if* $E = \mathcal{F}_q H^2(\mathbb{C}_+)$ *for a measurable function* q *on* $\mathbb{R}$ *with* $|q| = 1$ *a.e.*

Proof. Lemma 6.12 shows that E is 2 or 1-invariant if and only if its preimage $U_2^{-1}E \subset L^2(\mathbb{T})$ has the same property with respect to the shift operator on $L^2(\mathbb{R})$. The results thus follow by applying theorems 1.4, 1.5 and Theorem 6.5. □

Corollary 6.14. *Let* E *be a subspace of* $L^2(\mathbb{R})$.

(1) E is translation 2-invariant if and only if $E = \mathcal{F}\chi_\Sigma L^2(\mathbb{R})$ for a measurable subset $\Sigma \subset \mathbb{R}$.

(2) E is translation 1-invariant if and only if $E = \mathcal{F}qH^2(\mathbb{C}_+)$ for a measurable function q on $\mathbb{R}$ with $|q| = 1$ a.e.

Indeed, it suffices to use Theorem 6.13 and duality of Subsection 6.6.

Corollary 6.15. *(i) If $F \subset H^2(\mathbb{C}_+)$, then $\overline{span}_{H^2_+}\{e^{isx}F : s \geq 0\} = \Theta H^2(\mathbb{C}_+)$, where Θ is the g.c.d of the inner factors of $f \in F$.*

(ii) If $F \subset L^2(\mathbb{R}_+)$, then $\overline{span}_{L^2(\mathbb{R}_+)}\{\tau_s F : s \geq 0\} = \mathcal{F}(\Theta H^2(\mathbb{C}_+))$, where Θ is the g.c.d of the inner factors of $\mathcal{F}^{-1}f$, $f \in F$.

(iii) If $f \in L^2(\mathbb{R})$, then $\overline{span}_{L^2(\mathbb{R})}\{e^{isx}f : s \in \mathbb{R}\} = L^2(\mathbb{R})$ if and only if $f \neq 0$ a.e. on $\mathbb{R}$.

(iv) If $f \in L^2(\mathbb{R})$, then $\overline{span}_{L^2(\mathbb{R})}\{e^{isx}f : s \geq 0\} = L^2(\mathbb{R})$ if and only if $f \neq 0$ a.e. and
$$\int_{\mathbb{R}} (1 + x^2) \log |f| dx = -\infty$$

(v) If $f \in L^2(\mathbb{R})$, then $\overline{span}_{L^2(\mathbb{R})}\{\tau_s f : s \geq 0\} = L^2(\mathbb{R})$ if and only if $\mathcal{F}f \neq 0$ a.e. on $\mathbb{R}$

(vi) If $f \in L^2(\mathbb{R})$, then $\overline{span}_{L^2(\mathbb{R})}\{\tau_s f : s \geq 0\} = L^2(\mathbb{R})$ if and only if $\mathcal{F}f \neq 0$ a.e. and
$$\int_{\mathbb{R}} (1 + x^2) \log |\mathcal{F}f| dx = -\infty.$$

Indeed, it suffices to use Theorem 6.13 and Corollary 6.14 and the corresponding properties of z-invariant subspaces of $L^2(\mathbb{R})$.

6.7. Cauchy kernels and L^p- decomposition.

(a) Show that $H^p(\mathbb{C}_+) = \overline{\text{span}}_{L^2(\mathbb{R})}\left\{\dfrac{1}{x - \bar{\mu}} : \ \text{Im}\mu > 0\right\}$ for $1 \leq p \leq \infty$.
(Hint: Use $H^p(\mathbb{C}_+) = U_p H^p$ and solve $U_p f = \frac{1}{x-\bar{\mu}}$).

(b) Let $1 < p < \infty$. Show that $L^p(\mathbb{R}) = H^p(\mathbb{C}_+) \oplus H^p(\mathbb{C}_-)$, where $\oplus$ stands for the orthogonal sum for $p = 2$ and direct sum for $p \neq 2$.

(c) Let
$$Cf(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - z} dt, \quad z \in \mathbb{C} \setminus \mathbb{R}$$
be the Cauchy integral of $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then the followings are equivalent.

(a) $f \in H^p(\mathbb{C}_+)$.

(b) $Cf = f_*$, where f_* stands for the Poisson integral extension.

(c) $Cf(z) = 0$ for $\operatorname{Im} z < 0$.

Theorem 6.16. (*The Paley Wiener theorem*) *An entire function E is called of exponential type if*

$$\overline{\lim}_{|z|\to\infty} \frac{\log|E(z)|}{|z|} < \infty;$$

the limit itself is the type *of E. Let $\mathcal{E}_a$ = set of all entire functions of exponential type $\leq a$. For $a > 0$, show that the followings are equivalent.*

(i) $E \in \mathcal{E}_a$ and $E|_{\mathbb{R}} \in L^2(\mathbb{R})$.

(ii) There exists $f \in L^2(\mathbb{R})$ such that $\mathcal{F}f = E$ and $\operatorname{supp} f \in [-a, a]$.

(Hint: For (ii) $\Longrightarrow$ (i), estimate the exponential type of E applying the Cauchy inequality to the Fourier transform of f :

$$|E(z)| = \Big|\int_{-a}^{a} e^{-ixz} f(x)dx\Big| \leq \|f\|_2 \Big(\frac{e^{2a|\ Imz|}-1}{Imz}\Big)^{\frac{1}{2}} \leq (2a)^{\frac{1}{2}} e^{a|\ Imz|}.$$

Moreover, $\|E\|_2 = \|f\|_2$ by Plancherel's theorem:

(i) $\Longrightarrow$ (ii): First suppose that $E|_{\mathbb{R}} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then by Phragmèn-Lindelöf theorem $|E(z)| \leq \|E\|_\infty e^{a|\ Imz|}$, for $z \in \mathbb{C}$, implies

$$|E_\lambda(z)| = \frac{i\lambda}{z+i\lambda} e^{aiz} E(z) \in H^2(\mathbb{C}_+), \ \ \lambda > 0.$$

The Paley Wiener theorem 6.6 entails that $\mathcal{F}(E_\lambda) = 0$ a.e. on $(-\infty, 0)$ and hence $\mathcal{F}(e^{aiz}E) = 0$ on $(-\infty, a)$ (because $\lim\limits_{\lambda\to\infty} \|E_\lambda - e^{aiz}E\|_{L^2(\mathbb{R})} = 0$). Therefore, $\mathcal{F}(E) = \tau_a \mathcal{F}(e^{iaz}E) = 0$ a.e on $(-\infty, -a)$. Similarly $\mathcal{F}(E) = 0$ a.e. on $(a, \infty.)$ and we get (ii).

In general case, replace E by $E^\epsilon(z) = \int_{\mathbb{R}} E(z-t)\phi_\epsilon(t)dt$, where $\phi_\epsilon(t) = \epsilon^{-1}\phi(\frac{t}{\epsilon})$, $\phi \geq 0$ is compactly supported in $\mathbb{R}$. It is easy to see that $E^\epsilon \in \mathcal{E}_{a+\epsilon}$ and $\operatorname{supp}(E^\epsilon) \subset [-a-\epsilon, a+\epsilon]$, and we have $\lim\limits_{\epsilon\to 0} \|E^\epsilon - E\|_{L^2(\mathbb{R})} = 0$.

Question 6.17. (a) Show that $f \in H^2(\mathbb{C}_+)$ if and only if $f \in L^2(\mathbb{R})$ and $\mathcal{F}(f) = 0$ a.e. on $\mathbb{R}$.

(b) Find $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $L^2(\mathbb{R}) = \overline{\text{span}}_{L^2(\mathbb{R})}(\tau_s f : s \in \mathbb{R})$ and $L^1(\mathbb{R}) \neq \overline{\text{span}}_{L^1(\mathbb{R})}\{\tau_s f : s \in \mathbb{R}\}$ (Hint: Consider $f = \chi_{(a,b)}$)

(c) **Riesz Brother's theorem for** $\mathbb{R}$: Let μ be a complex Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} e^{ist} d\mu(t) = 0$ for all $s > 0$. Show that $\mu << m$.

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