APPLICATION OF HARMONIC ANALYSIS

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by

Abhishek Kumar
(Roll Number: 232123001)

to the

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI - 781039, INDIA

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CERTIFICATE

This is to certify that the work presented in this project report entitled **"Application of Harmonic Analysis"** has been carried out by **Abhishek Kumar** (Roll No. 232123001) under my supervision in partial fulfilment of the requirements for the degree of **Master of Science** in **Mathematics and Computing** at the Department of Mathematics, Indian Institute of Technology Guwahati.

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Dr. Rajesh Kumar Srivastava
Project Supervisor

ABSTRACT

Harmonic analysis provides a powerful language for understanding functions through their frequency content. This report surveys foundational tools of the subject—Fourier series, the Fourier transform, convolution, and basic integral transforms—and illustrates how these ideas lead to concrete methods for approximation and problem-solving. After discussing convergence and approximation phenomena for Fourier series, we develop the Fourier transform framework on classical function spaces and highlight inversion and energy identities (Parseval/Plancherel). The exposition emphasizes the guiding principle that many analytic and differential problems become simpler after passing to the frequency domain, and includes representative examples that demonstrate applications such as filtering, signal representation, and the solution of linear PDEs.

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Abhishek Kumar
Roll Number: 232123001

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Chapter 1

Introduction to Harmonic Function

1.1 Definition of Harmonic Function

Definition 1.1.1. Harmonic functions are real-valued functions that satisfy Laplace's equation in a given domain. These functions play a crucial role in various fields such as physics, engineering, and complex analysis, especially in the study of heat flow, electrostatics, and fluid dynamics.

A function $u(x, y)$ is said to be **harmonic** in a domain D if:

- All second-order partial derivatives of u exist and are continuous in D,
- And it satisfies the **Laplace equation**:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

1.2 Relationship with Analytic Functions

1.2.1 Analytic Functions and Harmonicity

Theorem 1.2.1. *Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function on an open set $\mathbb{D} \subset \mathbb{C}$. Then both u and v are harmonic in $\mathbb{D}$.*

Proof. Since $f(z)$ is analytic, it satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We now prove that u is harmonic:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}$$

Adding both:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \Rightarrow \Delta u = 0$$

Hence, u is harmonic.

Similarly, for v:

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial x \partial y}$$
$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

Adding both:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0 \Rightarrow \Delta v = 0$$

Therefore, both u and v are harmonic. □

1.3 Example

Let $f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$

Then,

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

Now, compute second derivatives:

$$u_{xx} = 2, \quad u_{yy} = -2 \Rightarrow u_{xx} + u_{yy} = 0$$

$$v_{xx} = 0, \quad v_{yy} = 0 \Rightarrow v_{xx} + v_{yy} = 0$$

So both u and v are harmonic functions.

1.4 Properties of Harmonic Functions

- The sum or difference of two harmonic functions is also harmonic.
- Constant functions are trivially harmonic.
- However, the product of two harmonic functions is not necessarily harmonic.

- The real and imaginary parts of an analytic function are harmonic.
- Harmonic functions are infinitely differentiable.
- They satisfy the mean value property and the maximum/minimum principle.

Chapter 2

Harmonic conjugate

2.1 Definition of Harmonic Conjugate

Let $u(x, y)$ and $v(x, y)$ be two real-valued functions that are harmonic in a region Ω. If these functions satisfy the Cauchy-Riemann equations in Ω, then $v(x, y)$ is said to be the harmonic conjugate of $u(x, y)$. In other words, if the function $f(z) = u(x, y) + iv(x, y)$ is complex differentiable (analytic) in Ω, then v is the harmonic conjugate of u, and the pair (u, v) forms the real and imaginary parts of an analytic function.

This conjugate relationship holds under the condition that both functions u and v are harmonic (i.e., they satisfy Laplace's equation) and their domain Ω does not exclude any essential singularities for $f(z)$.

Example:

Show that the function $u = 2xy+2x$ is **harmonic** and find the **corresponding conjugate harmonic function.**

Solution:

Given that $u(x, y) = 2xy + 2x$

$$\frac{\partial u}{\partial x} = 2y + 2, \quad \frac{\partial u}{\partial y} = 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow u(x, y) \text{ is a harmonic function}$$

To find $v(x,y)$:
Using Cauchy-Riemann equations:

$$\frac{\partial v}{\partial y} = 2y + 2, \quad -\frac{\partial v}{\partial x} = 2x$$

Integrating w.r.t. y, we get:

$$v = y^2 + 2y + g(x)$$

$$\Rightarrow -g'(x) = 2x \Rightarrow g(x) = -x^2 + c$$

$$\Rightarrow v(x,y) = y^2 + 2y - x^2 + c$$

$$\Rightarrow v(x,y) \text{ is the conjugate harmonic function of } u(x,y)$$

Verification: Show that $v(x,y)$ satisfies Laplace's equation.

$$\frac{\partial v}{\partial x} = -2x, \quad \frac{\partial^2 v}{\partial x^2} = -2$$

$$\frac{\partial v}{\partial y} = 2y + 2, \quad \frac{\partial^2 v}{\partial y^2} = 2$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -2 + 2 = 0$$

$$\Rightarrow v(x,y) \text{ is also a harmonic function}$$

Corollary 2.2.1

A harmonic function defined on a region has continuous partial derivatives of all orders.

Chapter 3

Poisson Integral Formula

Let f be analytic within and on a circle C defined by $|z| = R$, then

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \, d\phi$$

where $0 < r < R$.

Proof

Let $a = re^{i\theta}$, so a lies inside the circle C. By Cauchy's Integral Formula:

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} \, dz \quad (1)$$

The inverse point of a with respect to the circle $|z| = R$ is $\frac{R^2}{\bar{a}}$, which lies outside C. Since $\frac{f(z)}{z - \frac{R^2}{\bar{a}}}$ is analytic on and inside C, by Cauchy's Theorem:

$$0 = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \frac{R^2}{\bar{a}}} \, dz \quad (2)$$

Subtracting (2) from (1), we get:

$$f(a) = \frac{1}{2\pi i} \int_C \left(\frac{f(z)}{z - a} - \frac{f(z)}{z - \frac{R^2}{\bar{a}}} \right) dz$$

$$= \frac{1}{2\pi i} \int_C f(z) \left(\frac{1}{z - a} - \frac{1}{z - \frac{R^2}{\bar{a}}} \right) dz$$

$$= \frac{1}{2\pi i} \int_C f(z) \left(\frac{a - \frac{R^2}{\bar{a}}}{(z - a)(z - \frac{R^2}{\bar{a}})} \right) dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)(R^2 - a\bar{a})}{(z-a)(R^2 - \bar{z}a)}\, dz \quad (3)$$

Let $z = Re^{i\phi}$ and $a = re^{i\theta} \Rightarrow \bar{a} = re^{-i\theta}, \quad \bar{z} = Re^{-i\phi}$

Then,

$$\bar{z}a = Rre^{i(\theta-\phi)} \quad \Rightarrow \quad R^2 - \bar{z}a = R^2 - Rre^{i(\theta-\phi)}$$

$$= R^2 - Rr\cos(\theta - \phi) - iRr\sin(\theta - \phi)$$

and,

$$|z-a|^2 = R^2 - 2Rr\cos(\theta - \phi) + r^2$$

Using these, we arrive at:

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\, d\phi$$

Note on Harmonic Functions

If $f(z)$ is analytic, then $u = \mathrm{Re}(f(z))$ is harmonic.

By Cauchy's Mean Value Property:

$$\frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\, d\theta = u(x_0, y_0)$$

Mean Value Property

Suppose u is harmonic on $B_R(0)$ and $u(Re^{i\phi}) = f(\phi)$, where f is 2π-periodic and piecewise continuous. Then:

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi$$

Examples

1. $f(\theta) = 1 - \cos\theta + \sin\theta, \quad R = 1$

$$u(r, \theta) = 1 - r\cos\theta + r\sin\theta$$

2. $f(\theta) = \cos\theta - \frac{1}{2}\sin(2\theta), \quad R = 1$

$$u(r, \theta) = r\cos\theta - \frac{r^2}{2}\sin(2\theta)$$

3. $f(\theta) = 50\cos^2\theta, \quad R = 2$

   Using identity:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \Rightarrow f(\theta) = 25 + 25\cos(2\theta)$$

   Hence,

$$u(r,\theta) = 25 + 25r^2\cos(2\theta)$$

Theorem 3.0.1. ***(Mean Value Theorem).*** *Let $u : G \to \mathbb{R}$ be a harmonic function, and let $\overline{B}(a;r)$ be a closed disk contained in G. If γ is the circle $|z-a| = r$, then*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

Proof. Let D be a disk such that $\overline{B}(a;r) \subset D \subset G$, and let f be an analytic function on D such that $u = \mathrm{Re}(f)$.

By the Cauchy Integral Formula,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\, dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{(a+re^{i\theta})-a}\, ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\, d\theta. \end{aligned}$$

Comparing the real parts, we get:

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

□

Chapter 4

Harmonic Functions and Fourier Series

Let $0 \leq r < R$, and all θ. We have:

$$P(r, \theta) = \operatorname{Re}\left(\frac{R + re^{i\theta}}{R - re^{i\theta}}\right) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos(n\theta)$$

To see this, let $z = re^{i\theta}$. Then:

$$\frac{R+z}{R-z} = \left(\frac{R+z}{R}\right) \cdot \frac{1}{1 - \frac{z}{R}} = \left(1 + \frac{z}{R}\right) \cdot \sum_{n=0}^{\infty}\left(\frac{z}{R}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{z}{R}\right)^n + \sum_{n=1}^{\infty}\left(\frac{z}{R}\right)^n = 1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{R}\right)^n$$

Now using Euler's identity $z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$, we compare real parts:

$$\operatorname{Re}\left(\sum_{n=1}^{\infty}\left(\frac{z}{R}\right)^n\right) = \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos(n\theta)$$

Therefore,

$$P(r, \theta) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos(n\theta)$$

Fourier Coefficients

If f is piecewise continuous on $[0, 2\pi]$, then its Fourier coefficients are given by:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta, \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\, d\theta, \quad n = 1, 2, 3, \ldots$$

The coefficients a_n are called the **cosine Fourier coefficients** of f, and the b_n are the **sine Fourier coefficients** of f.

Theorem

Consider the Dirichlet problem with piecewise continuous boundary data f. Then the solution is

$$\psi(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos(n\theta) + b_n\sin(n\theta))$$

where a_n, b_n are the Fourier coefficients of f.

Proof:

We replace θ by $\theta - \phi$ in Poisson integral. In 1D form,

$$P(\theta, \phi) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

$$\psi(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} P(\theta, \phi) \cdot f(\phi)\, d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\phi) \cdot \left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos(n(\theta - \phi))\right] d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \int_0^{2\pi} f(\phi)\cos(n(\theta - \phi))\, d\phi$$

Since f is piecewise continuous, it is bounded on $[0, 2\pi]$. Let $A > 0$ be such that $|f(\phi)| \leq A$ for all ϕ. For fixed $0 \leq r < R$, we have

$$\left|\left(\frac{r}{R}\right)^n f(\phi)\cos(n(\theta - \phi))\right| \leq A\left(\frac{r}{R}\right)^n$$

So the series

$$\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n f(\phi)\cos(n(\theta - \phi))$$

converges uniformly in ϕ on the interval $[0, 2\pi]$ by the Weierstrass M-test.

Because

$$\sum_{n=1}^{\infty} A\left(\frac{r}{R}\right)^n < \infty$$

Integrate term by term, we get:

$$\psi(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cdot \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos(n(\theta-\phi))\,d\phi$$

$$= a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cdot \frac{1}{\pi}\int_0^{2\pi} f(\phi)\left(\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta)\right)d\phi$$

$$= a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n [a_n\cos(n\theta) + b_n\sin(n\theta)]$$

Proved

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