THE FATOU THEOREM

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CERTIFICATE

This is to certify that the work contained in this report entitled **"The Fatou theorem"** submitted by **Abhay Kumar** (**Roll No: 202123001.**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

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ABSTRACT

In this report we discus the existence of the nontangential limits of functions defined on open unit disc in the complex plane. The main result in that direction is the Fatou theorem. As a consequence enough harmonic functions are produced.

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Chapter 1

Introduction

We know that a continuous function on the unit circle $\Gamma = \{e^{it} : t \in \mathbb{R}\}$ gives a harmonic function by Poission Kernel. That is,

If $f \in \mathbb{C}(\Gamma)$, then $P * f$ is harmonic on the open unit disk $\mathbb{D}$. Conversly, a nonnegative harmonic function μ on $\mathbb{D}$ can be represented as Poission integral of a nonnegative finite Borel measure on Γ. And by Herglotz-Riesz representation theorem every analytic function on unit disc $\mathbb{D}$ can be represented by the Poission integral.

1.1 Absolute and continuous measure

Definition 1.1.1. [3] let μ be a positive measure on a σ-algebra $\mathbb{M}$, and let λ be an arbitrary measure on $\mathbb{M}$ (λ may be positive or complex). We say that λ is absolutely continuous with respect to μ (denoted by $\lambda \ll \mu$) if for every $E \in \mathbb{M}$ with $\mu(E) = 0$ implies $\lambda(E) = 0$.

And if there exists a set $A \subset \mathbb{M}$ such that $\lambda(E) = \lambda(A \cap E)$ for every $E \in \mathbb{M}$, we say that λ is concentrated on A. This is equivalent to say that $\lambda(E) = 0$ whenever $E \cap A = \emptyset$.

Suppose λ_1 and λ_2 are two measures on $\mathbb{M}$, and there exists a pair of disjoint sets A_1 and A_2 such that λ_i is concentrated on $A_i; i = 1, 2$. In this case we say that λ_1 is mutually singular to λ_2 and is denoted by $\lambda_1 \perp \lambda_2$.

Theorem 1.1.2. *(Radon-Nikodym decomposition) [1]*

Let μ be a positive σ-finite measure on a σ-algebra $\mathbb{M}$ in a set $\mathbb{X}$, and let λ be a complex measure on $\mathbb{M}$.

(i) Then there exists a unique pair of complex measures λ_a and λ_s on $\mathbb{M}$ such that

$$\lambda = \lambda_a + \lambda_s, \qquad \lambda_a \ll \mu, \qquad \lambda_s \perp \mu.$$

(ii) There is a unique $h \in L^1(\mu)$ such that $d\lambda_a = hd\mu$. Thus,

$$d\lambda = hd\mu + hd\mu_s.$$

1.2 Harmonic functions

Definition 1.2.1. A complex value function h on an open subset Ω of the the complex plane $\mathbb{C}$ is called harmonic function on Ω if $h \in \mathbb{C}^2$ and

$$\Delta h \equiv 0$$

on Ω. Here $\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$ is the Laplacian of h.

Harmonic functions arise in the study of analytic functions. If a function f is analytic on a region Ω, then by the Cauchy-Riemann equations each of the functions f, $\overline{f}$, Ref, Imf are harmonic on Ω.

Lemma 1.2.2. *Let Ω be a bounded open set in the complex plane with boundary $\partial\Omega$. Let h be a continuous complex-valued function on $\overline{\Omega} = \Omega \cup \partial\Omega$, which is harmonic on Ω. If $h|\partial\Omega \equiv 0$, then $h|\Omega \equiv 0$.*

1.3 The Poisson kernel

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane, and let $\Gamma = \{e^{it} : t \in \mathbb{R}\}$ be its boundary. The function

$$P(z, e^{it}) = Re\frac{e^{it} + z}{e^{it} - z} = \frac{1 - |z|^2}{|e^{it} - z|^2}$$

defined on $\mathbb{D} \times \Gamma$ is called the **Poisson kernel**. In polar coordinates, we have,

$$P(re^{i\theta}, e^{it}) = \frac{1 - r^2}{1 - 2rcos(\theta - t) + r^2} = \sum_{n=-\infty}^{\infty} r^{|j|} e^{ij(\theta - t)},$$

where the infinite series converges uniformly for every real values of θ, t and $0 \leq r \leq R$ for every $R \in (0, 1)$. For each fixed $e^{it} \in \Gamma$, the function $P(z, e^{it})$ is harmonic on $\mathbb{D}$.

And for every complex Borel measure μ on Γ, the function h defined by

$$h(z) = \int_{\Gamma} P(z, e^{it}) d\mu(e^{it}), \qquad z \in \mathbb{D},$$

is harmonic on $\mathbb{D}$.

Theorem 1.3.1. *For every continuous complex-valued function f on Γ there is a unique continuous function h on $\overline{\mathbb{D}} = \mathbb{D} \cup \Gamma$ such that $h|_{\Gamma} \equiv f$ and $h|\mathbb{D}$ is harmonic. The function h is given on $\mathbb{D}$ by*

$$h(z) = \int_{\Gamma} p(z, e^{it}) f(e^{it}) d\sigma(e^{it}), \qquad z \in \mathbb{D}. \tag{1.1}$$

In particular, for given continuous function h on $\overline{\mathbb{D}}$, whose restriction to $\mathbb{D}$ is harmonic, then $h|\mathbb{D}$ is of the form (1.1).

1.4 Properties of harmonic function

For any complex number a and a positive real number R , set $\mathbb{D}(a, R) = \{\, z : |z - a| < R \,\}$. And $\overline{\mathbb{D}}(a, R) = \{\, z : |z - a| \leq R \,\}$.

If h is continuous function on $\overline{\mathbb{D}}(a,R)$ and harmonic on $\mathbb{D}(a,R)$, then by theorem (1.3.1)

$$h(a+z) = \int_{\Gamma} P(z/R, e^{it}) h(a+Re^{it}) d\sigma(e^{it}), \qquad |z| < R. \tag{1.2}$$

Remarks:

(i) If h is real valued and harmonic function in a disk $\mathbb{D}(a,R)$, then $h = Re(f)$ for some function f which is analytic in $\mathbb{D}(a,R)$.

(ii) If h is harmonic function in a region Ω, then $h \in C^{\infty}(\Omega)$.

(iii) If $h = \lim_{n\to\infty} h_n$ uniformly on all compact subsets of a region Ω and each h_n is harmonic on Ω, then h is **harmonic** on Ω.

(iv) **(Mean value property)** If h is a harmonic function on a region Ω and $\overline{\mathbb{D}}(a,R) \subseteq \Omega$, then

$$h(a) = \frac{1}{2\pi} \int_0^{2\pi} h(a+Re^{it}) dt. \tag{1.3}$$

1.5 Weak compactness principle

Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of complex Borel measures on Γ such that $|\mu_n|(\Gamma) \leq M < \infty$ for all $n \in \{1,2,3...\}$, and for some constant M.
Then there exists a subsequence $\{\mu_{n_k}\}$ and a complex Borel measure μ on Γ such that $|\mu|(\Gamma) \leq M < \infty$, and

$$\lim_{k\to\infty} \int_{\Gamma} f d\mu_{n_k} = \int_{\Gamma} f d\mu \tag{1.4}$$

for every continuous complex-valued function f on Γ.

Theorem 1.5.1. *Every nonnegative harmonic function h on the unit disk $\mathbb{D}$ can be represented as*

$$h(z) = \int_{\Gamma} P(z, e^{it}) d\mu(e^{it}), \qquad z \in \mathbb{D}, \tag{1.5}$$

where μ is a finite nonnegative Borel measure on Γ.

Proof. We know for any complex number a and positive real number R. If h is continuous on $\overline{\mathbb{D}}(a,R)$ and harmonic on $\mathbb{D}(a,R)$, then

$$h(a+z)=\int_\Gamma P(z/R,e^{it})h(a+Re^{it})d\sigma(e^{it}), \qquad |z|<R. \tag{1.6}$$

Since here $\mathbb{D}$ is unit disk so R=1 and putting a=0 in the above equation we get

$$h(z)=\int_\Gamma P(z,e^{it})h(e^{it})d\sigma(e^{it}), \qquad z\in\mathbb{D}, \tag{1.7}$$

for each $r\in(0,1)$, we have

$$h(rz)=\int_\Gamma P(z,e^{it})h(re^{it})d\sigma(e^{it}), \qquad z\in\mathbb{D}. \tag{1.8}$$

Taking $d\mu_r=h(re^{it})d\sigma(e^{it})$ is nonnegative measure on Γ with total mass $h(0)$. Since $h(0)=\frac{1}{2\pi}\int_\Gamma p(re^{it})$ (by mean value property) therfore, (1.8) becomes

$$h(rz)=\int_\Gamma P(z,e^{it})d\mu_r(e^{it}), \qquad \forall z\in\mathbb{D}. \tag{1.9}$$

From weak compactness principle, we know there exists a sequence $r_n\uparrow 1$ and a finite nonnegative measure μ on Γ such that

$$\lim_{n\to\infty}\int_\Gamma f d\mu_{r_n}=\int_\Gamma f d\mu \tag{1.10}$$

for each continuous function f on Γ. By setting $r=r_n$ in 1.9 and $r_n\uparrow 1$ we have

$$\lim_{r_n\to 1}h(r_n z)=\lim_{r_n\to 1}\int_\Gamma P(z,e^{it})d\mu_{r_n}(e^{it})=\int_\Gamma P(z,e^{it})d\mu(e^{it}),\forall z\in\mathbb{D}. \tag{1.11}$$

Since h is continuous, it follows that

$$h(z)=\int_\Gamma P(z,e^{it})d\mu(e^{it}), \qquad \forall z\in\mathbb{D}. \tag{1.12}$$

$\square$

1.6 Herglotz and Riesz representation theorem.

Let f be a analytic function such that $Ref \geq 0$ on $\mathbb{D}$. Then

$$f(z) = \int_{\Gamma} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(e^{i\theta}) + ic, \qquad z \in \mathbb{D}, \tag{1.13}$$

for some finite non negative Borel measure μ on Γ and some constant c.

Definition 1.6.1. Let $h^1(\mathbb{D})$ be the class of all real-valued harmonic function h on the unit disk $\mathbb{D}$ such that

$$\sup_{0<r<1} \int_{\Gamma} |h(re^{i\theta})| d\sigma(e^{i\theta}) < \infty. \tag{1.14}$$

Theorem 1.6.2. *Let h be real valued harmonic function on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $h \in h^1(\mathbb{D})$*;*

(ii) $h = h^+ - h^-$ *where* h^+ *and* h^- *are nonnegative harmonic function on* $\mathbb{D}$*;*

(iii) there exists a real-valued Borel measure μ *on* Γ *such that*

$$h(z) = \int_{\Gamma} P(z, e^{i\theta}) \, d\mu(e^{i\theta}), \qquad z \in \mathbb{D}; \tag{1.15}$$

(iv) there exists a nonnegative harmonic function k *on* $\mathbb{D}$ *such that*

$$|h(z)| \leq k(z), \qquad z \in \mathbb{D}. \tag{1.16}$$

1.7 Non tangential limit

Definition 1.7.1. Let h be complex valued function on $\mathbb{D}$, and let $e^{i\tau}$ be a point of Γ. We write "$Lim_{z \to e^{i\tau}} h(z) = A$ nontangentially" if for every open triangular sector S in D with vertex at $e^{i\tau}$, $h(z) \to A$ as $z \to e^{i\tau}$ whithin S.

We say that "$lim_{z\to e^{it}} h(z) = f(e^{it})$ nontangentially a.e." if there exist a Borel set $N \subseteq \Gamma$ with $\sigma(N) = 0$ such that $lim_{z\to e^{it}} h(z) = f(e^{it})$ nontangentially for each $e^{it} \in \Gamma\backslash N$.

1.8 Fatous's theorem.

[2] Let μ be a complex Borel measure on Γ with Lebesgue decomposition

$$d\mu = f d\sigma + d\mu_s, \tag{1.17}$$

where μ_s is singular with respect to σ. If

$$h(z) = \int_\Gamma P(z, e^{it}) d\mu(e^{it}), \qquad z \in \mathbb{D}, \tag{1.18}$$

then

$$\lim_{z\to e^{it}} h(z) = f(e^{it}) \tag{1.19}$$

nontangentially σ-a.e. on Γ.

Proof. Let $\alpha(t)$ be a distribution function for μ. To prove the result, it is sufficien to show that (1.19) holds nontangentially at any point e^{it} such that $\alpha^{'}(t)$ exists and equal to $f(e^{it})/2\pi$. Whithout loss of generally, we can assume that $t = 0$ and $\alpha(0) = 0$. Thus, from assumption that

$$\alpha'(0) = \lim_{t\to 0} \frac{\alpha(t)}{t}$$

exists, we prove that $h(z) \to 2\pi\alpha'(0)$ nontangentially as $z \to 1$. Fix a sector in the unit disk with vertex at 1, say

$$S = \{z : |y| < K(1-x), c < x < 1\} \tag{1.20}$$

where $K > 0$ and c is a positive and $0 < c < 1$ but near to 1.

Let $\epsilon > 0$ be given. We have to show that there exists a $\delta > 0$ such that the inequality $|h(z) - 2\pi\alpha'(0)| < \epsilon$ holds, whenever $z \in S$ and $|z-1| < \delta$. To this we write

$$\begin{aligned}
h(z) - 2\pi\alpha'(0) &= \int_{-\pi}^{\pi} P(z, e^{it})\, d\alpha(t) - \int_{-\pi}^{\pi} d(\alpha'(0)t) \\
&= \int_{-\pi}^{\pi} P(z, e^{it})\, d[\alpha(t) - \alpha'(0)t] \\
&= \{ P(z, e^{it})[\alpha(t) - \alpha'(0)t]\}_{t=-\pi}^{\pi} \\
&- \int_{-\pi}^{\pi} [\alpha(t) - \alpha'(0)t] \frac{\partial}{\partial t} P(z, e^{it}) dt \\
&= \frac{1-|z|^2}{|1+z|^2} [\alpha(\pi) - \alpha(-\pi) - 2\pi\alpha'(0)] \\
&- \int_{|t|\le\xi} [\alpha(t) - \alpha'(0)t] \frac{\partial}{\partial t} P(z, e^{it}) dt \\
&- \int_{\xi<|t|\le\pi} [\alpha(t) - \alpha'(0)t] \frac{\partial}{\partial t} P(z, e^{it}) dt \\
&= I + II + III.
\end{aligned}$$

where $\xi \in (0, \pi)$ such that

$$\left| \frac{\alpha(t)}{t} - \alpha'(0) \right| < \epsilon/M, \qquad M = 3(2\pi + 16K), \tag{1.21}$$

for $0 < |t| \le \xi$. Since $I \to 0$ as $z \to 1$, we can choose $\delta_1 > 0$ such that $|I| < \epsilon/3$ if $z \in S, |z-1| < \delta_1$.

In the integrand of the third term (III)

$$\frac{\partial}{\partial t} P(z, e^{it}) = \frac{\partial}{\partial t} Re \frac{e^{it} + z}{e^{it} - z} = Re \frac{-2ize^{it}}{(e^{it} - z)^2},$$

tends to 0 uniformly for $\xi < |t| \le \pi$ as $z \to 1$.
Hence we may choose $\delta_3 > 0$ such that $|III| < \epsilon/3$ if $z \in S$, $|z-1| < \delta_3$. Now

we have to estimate only second (II) integrand. For any $z \in S$,

$$\begin{aligned}
|II| &= \left| \int_{-\xi}^{\xi} \left[\frac{\alpha(t)}{t} - \alpha'(0) \right] t \frac{\partial}{\partial t} P(z, e^{it}) dt \right| \\
&\leq \frac{\epsilon}{M} \int_{-\xi}^{\xi} \left| t \frac{\partial}{\partial t} P(z, e^{it}) \right| dt \qquad (from\ 1.21) \\
&= \frac{\epsilon}{M} \int_{-\xi}^{\xi} \left| \frac{t(1-r^2)2r\sin(t-\theta)}{(1-2r\cos(\theta-t)+r^2)^2} \right| dt,
\end{aligned}$$

where $z = re^{i\theta}$. If we replace t by $\theta + t$ in the last integral and asuume that c in (1.20) is sufficiently near to 1, then we observe that the new range of integration is contained in $[-\pi, \pi]$. Hence

$$\begin{aligned}
|II| &\leq \frac{\epsilon}{M} \int_{-\pi}^{\pi} \left| (t+\theta) \frac{(1-r^2)2r\sin(t)}{(1-2r\cos(t)+r^2)^2} \right| dt \\
&\leq \frac{\epsilon}{M} \int_{-\pi}^{\pi} \left| t \frac{\partial}{\partial t} P(r, e^{it}) \right| dt + \frac{\epsilon}{M} |\theta| \int_{-\pi}^{\pi} \left| \frac{\partial}{\partial t} P(r, e^{it}) \right| dt.
\end{aligned}$$

Now

$$\begin{aligned}
\int_{-\pi}^{\pi} \left| t \frac{\partial}{\partial t} P(r, e^{it}) \right| dt &= \int_{-\pi}^{\pi} t \frac{\partial}{\partial t} P(r, e^{it}) dt \\
&= [tP(r, e^{it})]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} P(r, e^{it}) dt \\
&= 2\pi \frac{1-r}{1+r} - 2\pi \\
&< 2\pi.
\end{aligned}$$

and

$$\begin{aligned}
|\theta| \int_{-\pi}^{\pi} \left| \frac{\partial}{\partial t} P(r, e^{it}) \right| dt &= 2|\theta| \int_{0}^{\pi} \left| \frac{\partial}{\partial t} P(r, e^{it}) \right| dt \\
&= -2|\theta| [P(r, e^{it})]_0^{\pi} = \frac{8r|\theta|}{1-r^2} \leq \frac{8r|\theta|}{1-r}.
\end{aligned}$$

By (1.20), for $z = re^{i\theta} \in S, K(1 - r\cos\theta) > |r\sin\theta|$, so

$$K(1-r) + Kr(1-\cos\theta) > r|\sin\theta|$$

and hence

$$\begin{aligned} K(1-r) &> r|\sin\theta| - Kr(1-\cos\theta) \\ &= r|\theta|\left(\frac{\sin\theta}{\theta} - K\frac{1-\cos\theta}{\theta^2}|\theta|\right), \end{aligned}$$

where the functions on the right are defined by continuity, when $\theta = 0$. Choose $\delta_2 > 0$ so that $z \in S$, $|z-1| < \delta_2$ implies that

$$\begin{aligned} \frac{\sin\theta}{\theta} - K\frac{1-\cos\theta}{\theta^2}|\theta| &> \frac{1}{2} \\ \Longrightarrow K(1-r) &> \frac{1}{2}r|\theta| \\ \Longrightarrow 2K &> \frac{r|\theta|}{1-r}. \end{aligned}$$

Then for such z, we get

$$\begin{aligned} \frac{8r|\theta|}{1-r} &< 16K \\ \Longrightarrow |\theta|\int_{-\pi}^{\pi}\left|\frac{\partial}{\partial t}P(r,e^{it})\right|dt &< 16K, \end{aligned}$$

After combining both inequalities, we get

$$|II| < \frac{\epsilon}{M}(2\pi + 16K) = \frac{\epsilon}{3}.$$

Setting $\delta = min(\delta_1, \delta_2, \delta_3)$, we obtain

$$|h(z) - 2\pi\alpha'(0)| \leq |I| + |II| + |III| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for all $z \in S$ such that $|z-1| < \delta$, as was to be proved. $\square$

Corollary 1.8.1. *If f is analytic and bounded for the open unit disc $\mathbb{D}$, then*

$$\tilde{f}(e^{it}) = \lim_{z \to e^{it}} f(z) \tag{1.22}$$

exists nontangentially σ-a.e. on Γ.

The function $\tilde{f}$ is called boundary function of f.

Proof. Note that

$$f(z) = f * p(z) = \int_{\Gamma} f(e^{it}) p(z, e^{it}) d\sigma(e^{it}). \tag{1.23}$$

Hence, the result for corollary follows from 1.18 □

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