# ME 101: Engineering Mechanics

**Rajib Kumar Bhattacharjya**

Department of Civil Engineering

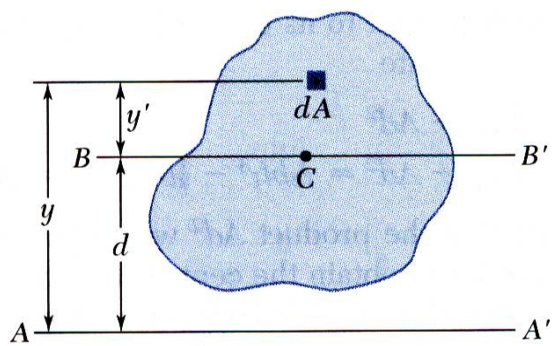
Indian Institute of Technology Guwahati

M Block : Room No 005 : Tel: 2428

www.iitg.ernet.in/rkbc

# Area Moments of Inertia

## Parallel Axis Theorem




- Consider moment of inertia $I$ of an area $A$ with respect to the axis $AA'$

$$I = \int y^2 dA$$

- The axis $BB'$ passes through the area centroid and is called a *centroidal axis.*

$$I = \int y^2 dA = \int (y' + d)^2 dA$$

$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

- Second term = 0 since centroid lies on BB' ($\int y' dA = y_c A$, and $y_c = 0$

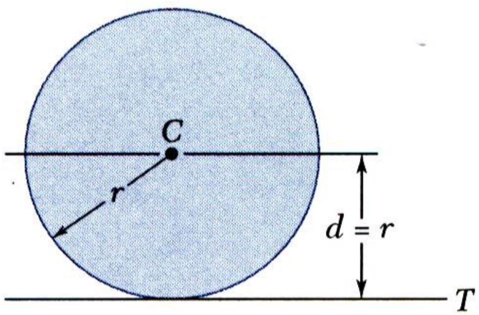
$$I = \bar{I} + Ad^2 \quad \textit{Parallel Axis theorem}$$

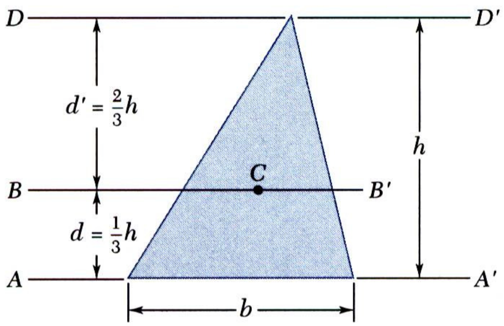
*Parallel Axis theorem*:
MI @ any axis =
MI @ centroidal axis + $Ad^2$

*The two axes should be parallel to each other.*

# Area Moments of Inertia

## Parallel Axis Theorem




- Moment of inertia $I_T$ of a circular area with respect to a tangent to the circle,

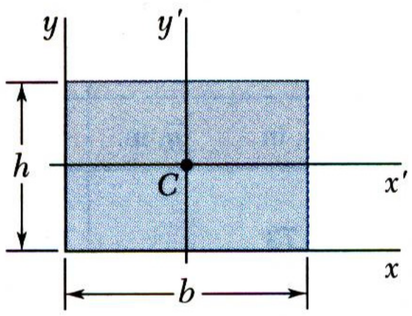
$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + \left(\pi r^2\right)r^2$$
$$= \tfrac{5}{4}\pi r^4$$

- Moment of inertia of a triangle with respect to a centroidal axis,

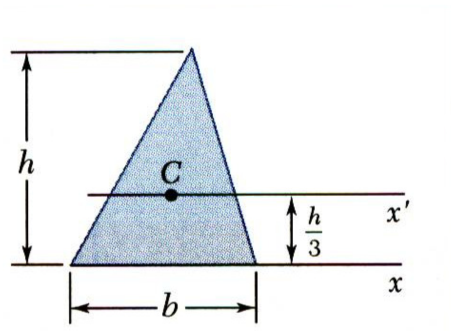
$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$
$$I_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh\left(\tfrac{1}{3}h\right)^2$$
$$= \tfrac{1}{36}bh^3$$

- The moment of inertia of a composite area $A$ about a given axis is obtained by adding the moments of inertia of the component areas $A_1$, $A_2$, $A_3$, ... , with respect to the same axis.

# Area Moments of Inertia: Standard MIs

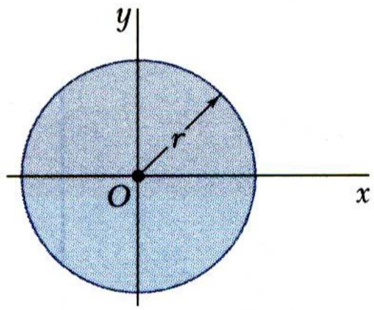



| | Answer |
|---|---|
| Moment of inertia about $x$-axis | $I_x = \frac{1}{3} b h^3$ |
| Moment of inertia about $y$-axis | $I_y = \frac{1}{3} b^3 h$ |
| Moment of inertia about $x'$-axis | $I_{x'} = \frac{1}{12} b h^3$ |
| Moment of inertia about $y'$-axis | $I_{y'} = \frac{1}{12} b^3 h$ |
| Moment of inertia about $z$-axis passing through C | $I_C = \frac{1}{12} bh(b^2 + h^2)$ |



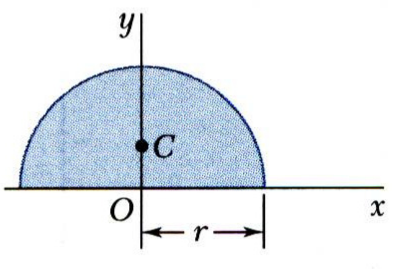

Moment of inertia about $x$-axis $$I_x = \frac{1}{12}bh^3$$

Moment of inertia about $x'$-axis $$I_{x'} = \frac{1}{36}bh^3$$



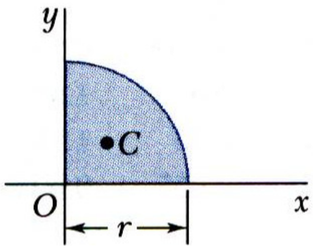

Moment of inertia about $x'$-axis $I_x = \dfrac{1}{4}\pi r^4$

Moment of inertia about $z$-axis passing through O $I_0 = \dfrac{1}{2}\pi r^4$



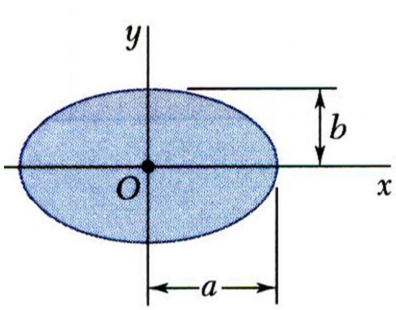

Moment of inertia about $x'$-axis $I_x = I_y = \frac{1}{8}\pi r^4$

Moment of inertia about $z$-axis passing through O $I_0 = \frac{1}{4}\pi r^4$




Moment of inertia about $x'$-axis $I_x = I_y = \frac{1}{16}\pi r^4$

Moment of inertia about $z$-axis passing through O $I_0 = \frac{1}{8}\pi r^4$




Moment of inertia about $x$-axis $$I_x = \frac{1}{4}\pi a b^3$$

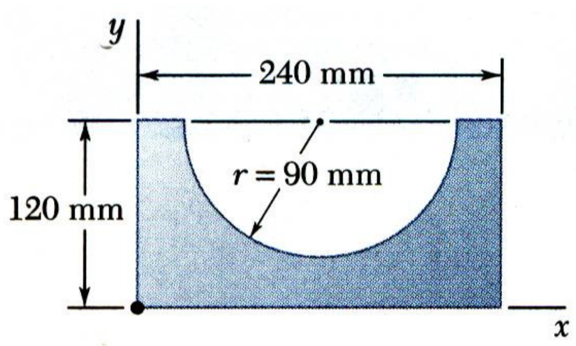
Moment of inertia about $y$-axis $$I_y = \frac{1}{4}\pi a^3 b$$

Moment of inertia about $z$-axis passing through O $$I_0 = \frac{1}{4}\pi ab(a^2 + b^2)$$

# Area Moments of Inertia

## Example:

Determine the moment of inertia of the shaded area with respect to the *x* axis.

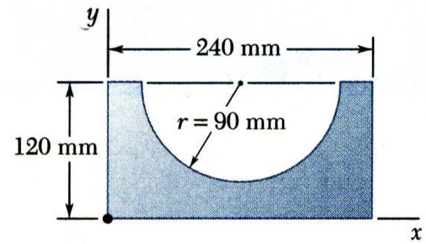


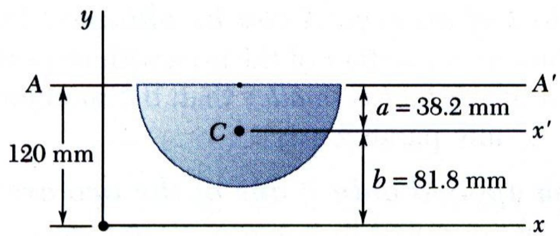

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.
- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.

# Area Moments of Inertia

## Example: Solution




$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2\,mm$$

$$b = 120 - a = 81.8\,mm$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$

$$= 12.72\times10^3\,mm^2$$

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the $x$ axis.

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120)^3 = 138.2\times10^6\,mm^4$$

Half-circle:

moment of inertia with respect to $AA'$,

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76\times10^6\,\mathrm{mm}^4$$

Moment of inertia with respect to $x'$,

$$\bar{I}_{x'} = I_{AA'} - Aa^2 = (25.76\times10^6) - (12.72\times10^3)(38.2)^2$$

$$= 7.20\times10^6\,mm^4$$

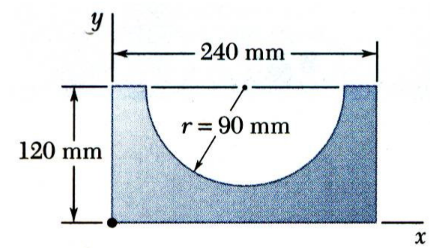
moment of inertia with respect to $x$,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20\times10^6 + (12.72\times10^3)(81.8)^2$$

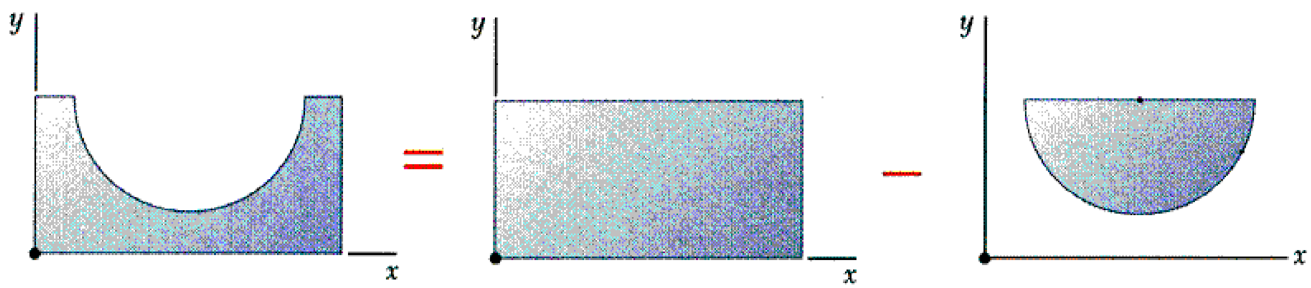
$$= 92.3\times10^6\,mm^4$$

# Area Moments of Inertia

## Example: Solution




- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.

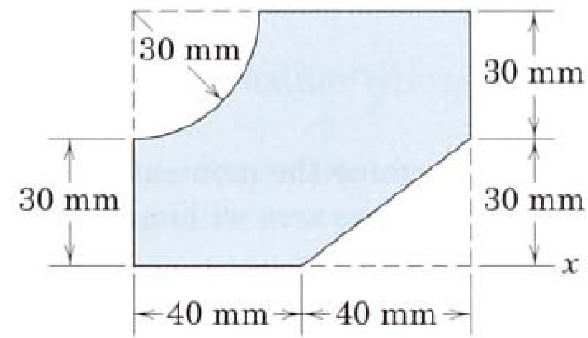



$$I_x \quad = \quad 138.2\times10^6\,\text{mm}^4 \quad - \quad 92.3\times10^6\,\text{mm}^4$$

$$\boxed{I_x = 45.9\times10^6\,\text{mm}^4}$$

## Consider area (1)

$$I_x = \frac{1}{3}bh^3 = \frac{1}{3} \times 80 \times 60^3 = 5.76 \times 10^6\ mm^4$$

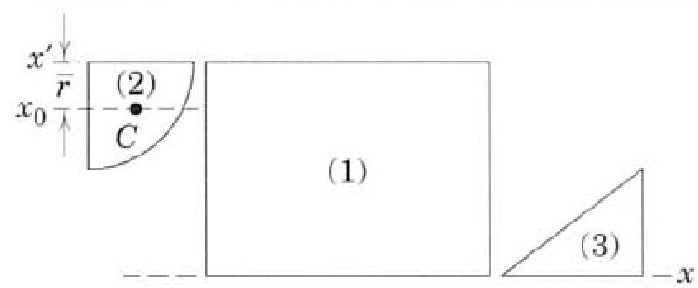



## Consider area (2)

$$I_{x'} = \frac{1}{4}\left(\frac{\pi r^4}{4}\right) = \frac{\pi}{16}(30)^4 = 0.1590 \times 10^6\ mm^4$$

$$\overline{I_x} = 0.1590 \times 10^6 - \frac{\pi}{4}(30)^2 \times (12.73)^2 = 0.0445 \times 10^6\ mm^4$$

$$I_x = 0.0445 \times 10^6 + \frac{\pi}{4}(30)^2(60 - 12.73) = 1.624 \times 10^6\ mm^4$$




## Consider area (3)

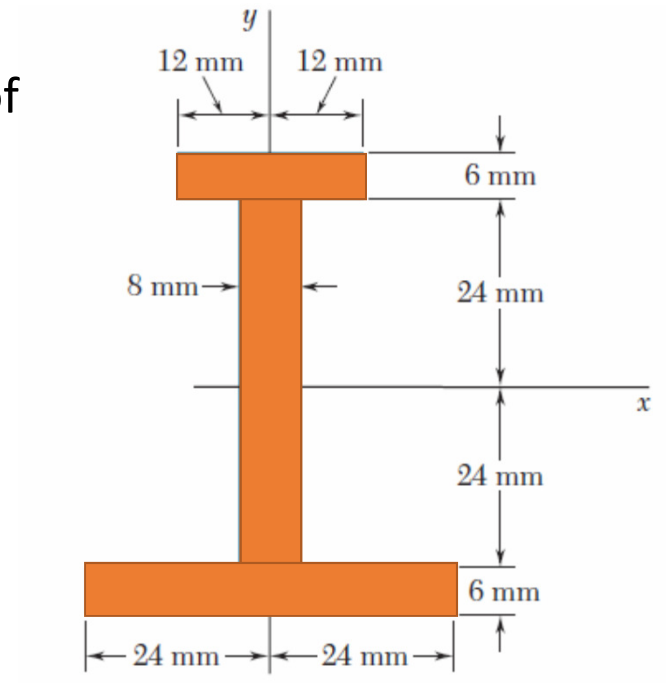
$$I_x = \frac{1}{12}bh^3 = \frac{1}{12} \times 40 \times 30^3 = 0.09 \times 10^6\ mm^4$$

$$I_x = 5.76 \times 10^6 - 1.624 \times 10^6 - 0.09 \times 10^6 = \mathbf{4.05 \times 10^6\ mm^4}$$

$$A = 60 \times 80 - \frac{1}{4}\pi(30)^2 - \frac{1}{2}40 \times 30 = 3490mm^2$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{4.05 \times 10^6}{3490}} = \mathbf{34.00mm}$$

Determine the moment of inertia and the radius of gyration of the area shown in the fig.




$$\overline{I1_x} = \frac{1}{12}bd^3 = \frac{1}{12} \times 24 \times 6^3 = 432\ mm^4$$

$$\overline{I2_x} = \frac{1}{12}bd^3 = \frac{1}{12} \times 8 \times 48^3 = 73728\ mm^4$$

$$\overline{I3_x} = \frac{1}{12}bd^3 = \frac{1}{12} \times 48 \times 6^3 = 864\ mm^4$$

$$I1_x = \overline{I1_x} + Ah^2 = 432 + 24 \times 6 \times (24+3)^2 = 105408\ mm^4$$

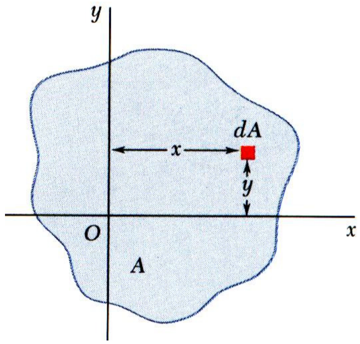
$$I3_x = \overline{I3_x} + Ah^2 = 864 + 48 \times 6 \times (24+3)^2 = 210816\ mm^4$$

$$I_x = 105408 + 73728 + 210816 = 390 \times 10^3 mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390 \times 10^3}{816}} = \mathbf{21.9\ mm}$$

# Area Moments of Inertia

Products of Inertia: for problems involving unsymmetrical cross-sections and in calculation of MI about rotated axes. It may be +ve, -ve, or zero

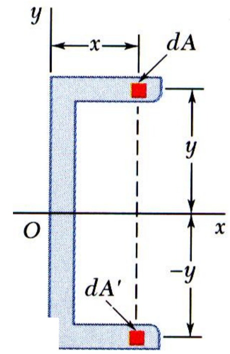



- Product of Inertia of area $A$ w.r.t. $x$-$y$ axes:

  $$I_{xy} = \int xy\, dA$$

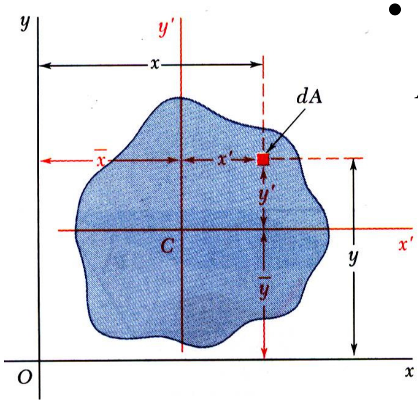
  $x$ and $y$ are the coordinates of the element of area $dA$=$xy$

- When the $x$ axis, the $y$ axis, or both are an axis of symmetry, the product of inertia is zero.

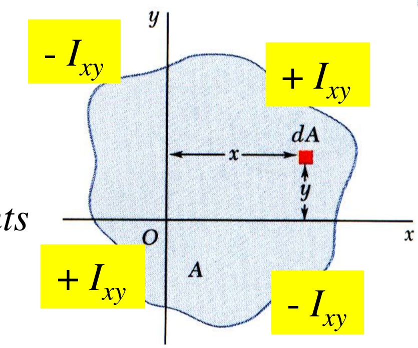



- Parallel axis theorem for products of inertia:

$$I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A$$




*Quadrants*

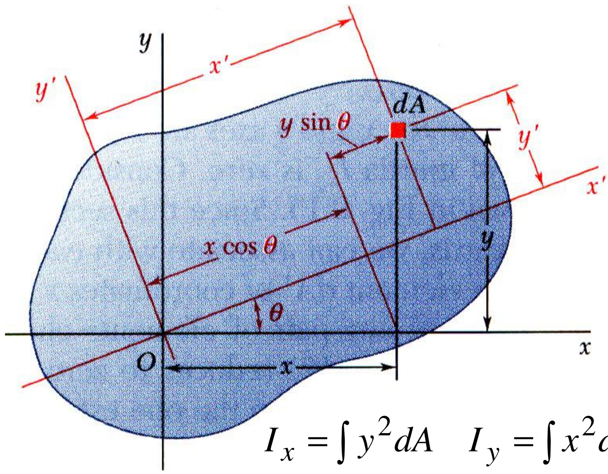


# Area Moments of Inertia

## Rotation of Axes

Product of inertia is useful in calculating MI @ inclined axes.

→ Determination of axes about which the MI is a maximum and a minimum




$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

$$I_{xy} = \int xy\, dA$$

Moments and product of inertia w.r.t. new axes $x'$ and $y'$?

Note: $x' = x\cos\theta + y\sin\theta$

$y' = y\cos\theta - x\sin\theta$

$$I_{x'} = \int y'^2 dA = \int (y\cos\theta - x\sin\theta)^2 dA$$

$$I_{y'} = \int x'^2 dA = \int (x\cos\theta + y\sin\theta)^2 dA$$

$$I_{x'y'} = \int x'y' dA = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta) dA$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin\theta\cos\theta = 1/2\sin 2\theta \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta$$

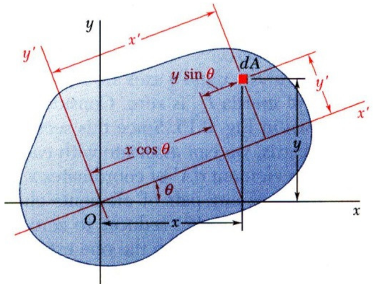
$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

# Area Moments of Inertia

## Rotation of Axes




$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

Adding first two eqns:

$I_{x'} + I_{y'} = I_x + I_y = I_z$ → The Polar MI @ O

Angle which makes $I_{x'}$ and $I_{y'}$ either max or min can be found by setting the derivative of either $I_{x'}$ or $I_{y'}$ w.r.t. θ equal to zero:

$$\frac{dI_{x'}}{d\theta} = \left(I_y - I_x\right)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

Denoting this critical angle by $\alpha$

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$$

→ two values of $2\alpha$ which differ by $\pi$ since $\tan 2\alpha = \tan(2\alpha+\pi)$

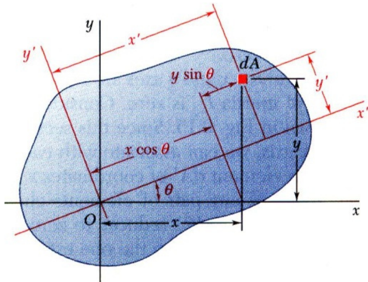
→ two solutions for $\alpha$ will differ by $\pi/2$

→ one value of $\alpha$ will define the axis of maximum MI and the other defines the axis of minimum MI

→ These two rectangular axes are called the principal axes of inertia

# Area Moments of Inertia

## Rotation of Axes




$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x} \Rightarrow \sin 2\alpha = \cos 2\alpha \frac{2I_{xy}}{I_y - I_x}$$

Substituting in the third eqn for critical value of 2θ: $I_{x'y'} = 0$

→ Product of Inertia $I_{x'y'}$ is zero for the Principal Axes of inertia

Substituting sin2$\alpha$ and cos2$\alpha$ in first two eqns for **Principal Moments of Inertia**:

$$I_{max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4I_{xy}^2}$$

$$I_{min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4I_{xy}^2}$$

$$I_{xy@\alpha} = 0$$

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} cos2\theta - I_{xy} sin2\theta \quad \Rightarrow \quad I_{x'} - \frac{I_x + I_y}{2} = \frac{I_x - I_y}{2} cos2\theta - I_{xy} sin2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2} sin2\theta + I_{xy} cos2\theta$$

Squaring both the equation and adding

$$\left(I_{x'} - \frac{I_x + I_y}{2}\right)^2 + \left(I_{x'y'}\right)^2 = \left(\frac{I_x - I_y}{2} cos2\theta - I_{xy} sin2\theta\right)^2 + \left(\frac{I_x - I_y}{2} sin2\theta + I_{xy} cos2\theta\right)^2$$

$$\left(I_{x'} - \frac{I_x + I_y}{2}\right)^2 + \left(I_{x'y'}\right)^2 = \left(\frac{I_x - I_y}{2}\right)^2 cos^2 2\theta - 2\,\frac{I_x - I_y}{2} I_{xy}\, cos2\theta sin2\theta + \left(I_{xy}\right)^2 sin^2 2\theta$$

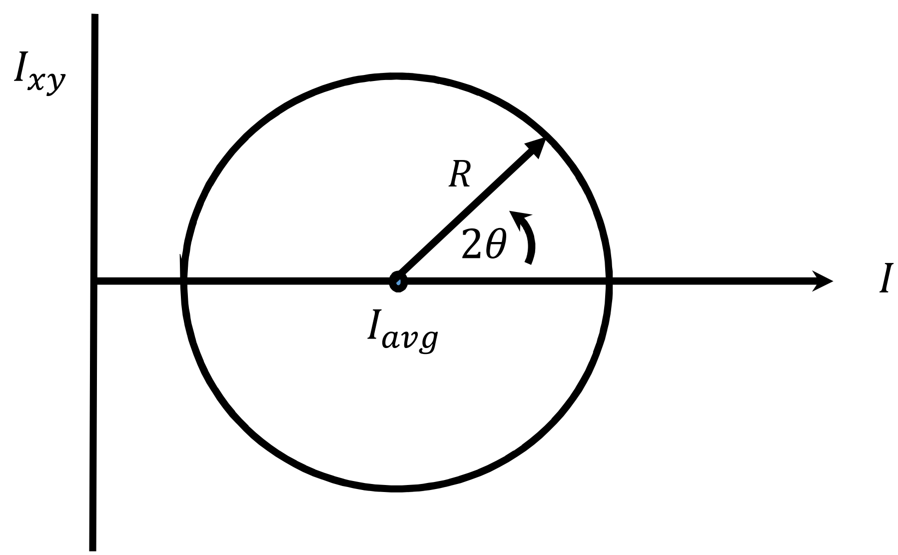
$$+\left(\frac{I_x - I_y}{2}\right)^2 sin^2 2\theta + 2\,\frac{I_x - I_y}{2} I_{xy}\, cos2\theta sin2\theta + \left(I_{xy}\right)^2 sin^2 2\theta$$

$$\left(I_{x'} - \frac{I_x + I_y}{2}\right)^2 + \left(I_{x'y'}\right)^2 = \left(\frac{I_x - I_y}{2}\right)^2 + \left(I_{xy}\right)^2$$

$$\left(I_{x'} - \frac{I_x + I_y}{2}\right)^2 + \left(I_{x'y'}\right)^2 = \left(\frac{I_x - I_y}{2}\right)^2 + \left(I_{xy}\right)^2$$

Defining $\frac{I_x + I_y}{2} = I_{avg}$ And $R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(I_{xy}\right)^2}$

$\left(I_{x'} - I_{avg}\right)^2 + \left(I_{x'y'}\right)^2 = R^2$ Which is a equation of circle with center $\left(I_{avg}, 0\right)$ and radius $R$



# Area Moments of Inertia

Mohr's Circle of Inertia: Following relations can be represented graphically by a diagram called Mohr's Circle

For given values of $I_x$, $I_y$, & $I_{xy}$, corresponding values of $I_{x'}$, $I_{y'}$, & $I_{x'y'}$ may be determined from the diagram for any desired angle θ.

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$$

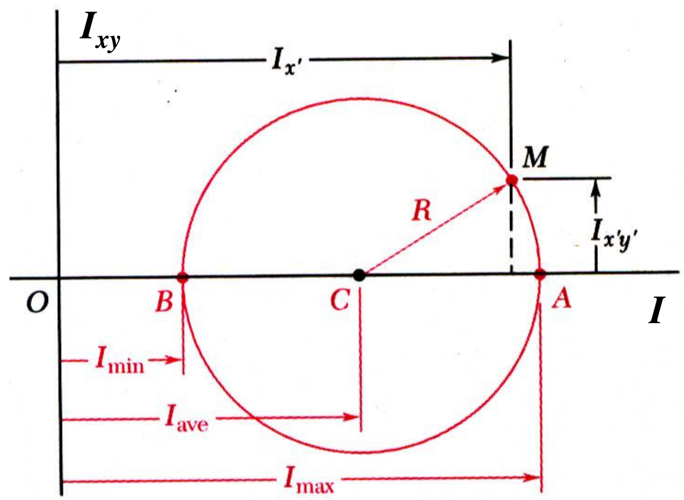
$$I_{\max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$

$$I_{\min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$

$$I_{xy@\alpha} = 0$$

$$(I_{x'} - I_{ave})^2 + I_{x'y'}^2 = R^2$$

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

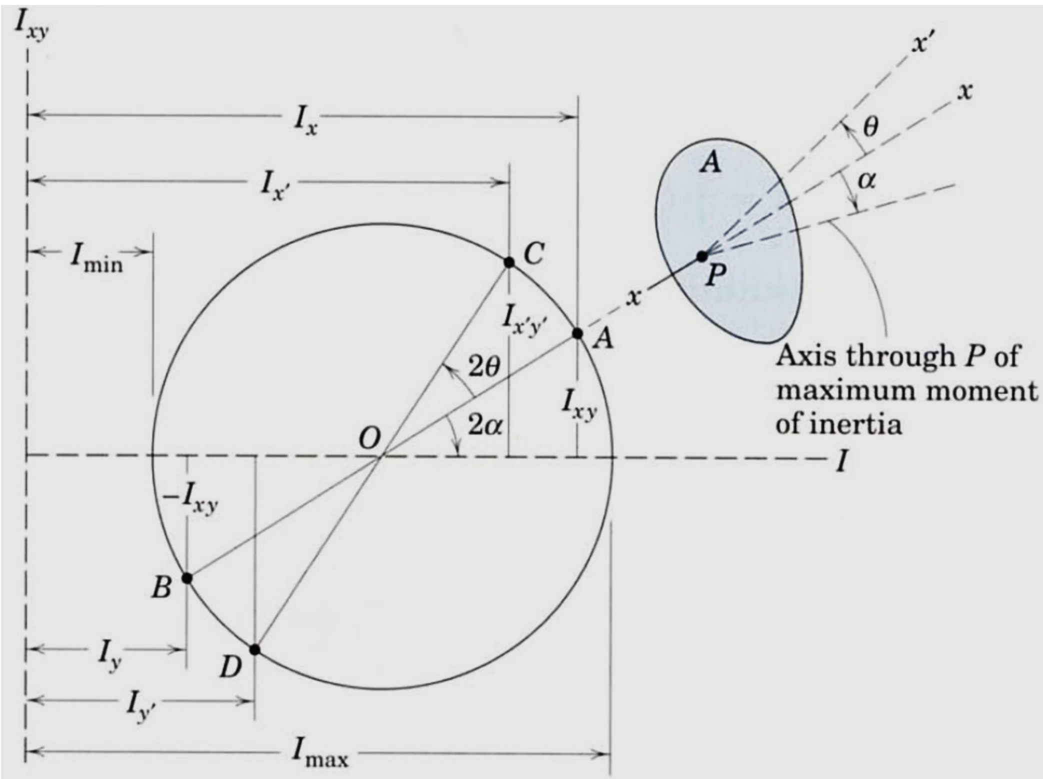



- At the points $A$ and $B$, $I_{x'y'} = 0$ and $I_{x'}$ is a maximum and minimum, respectively.

$$I_{\max,\min} = I_{ave} \pm R$$

# Area Moments of Inertia

## Mohr’s Circle of Inertia: **Construction**




$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$$

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$I_{max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4I_{xy}^2}$$

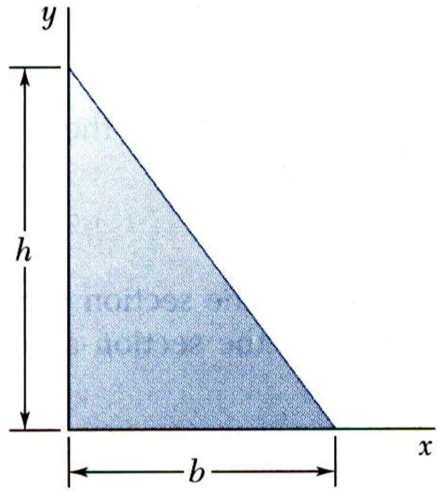
$$I_{min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4I_{xy}^2}$$

$$I_{xy@\alpha} = 0$$

Choose horz axis → MI
Choose vert axis → PI
Point A – known $\{I_x, I_{xy}\}$
Point B – known $\{I_y, -I_{xy}\}$
Circle with dia AB
Angle $\alpha$ for Area
→ Angle $2\alpha$ to horz (same sense) → $I_{max}$, $I_{min}$
Angle $x$ to $x' = \theta$
→ Angle OA to OC = $2\theta$
→ Same sense
Point C → $I_{x'}$, $I_{x'y'}$
Point D → $I_{y'}$

# Area Moments of Inertia

## Example: Product of Inertia



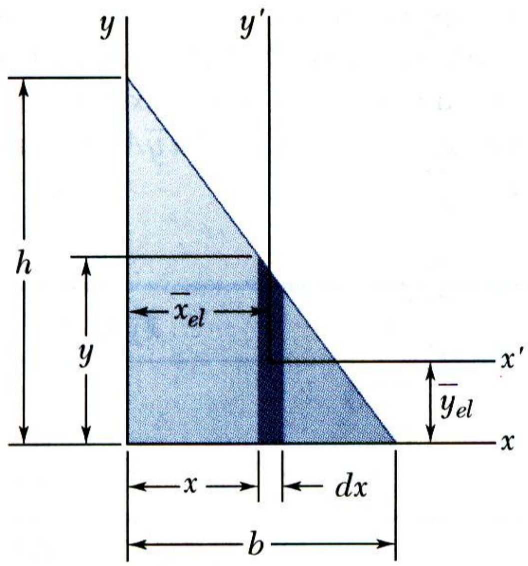

Determine the product of inertia of the right triangle *(a)* with respect to the $x$ and $y$ axes and *(b)* with respect to centroidal axes parallel to the $x$ and $y$ axes.

SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips
- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

# Area Moments of Inertia

## Examples




<u>SOLUTION</u>:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

$$y = h\left(1 - \frac{x}{b}\right) \qquad dA = y\,dx = h\left(1 - \frac{x}{b}\right)dx$$

$$\bar{x}_{el} = x \qquad \bar{y}_{el} = \tfrac{1}{2}y = \tfrac{1}{2}h\left(1 - \frac{x}{b}\right)$$

Integrating $dI_x$ from $x = 0$ to $x = b$,

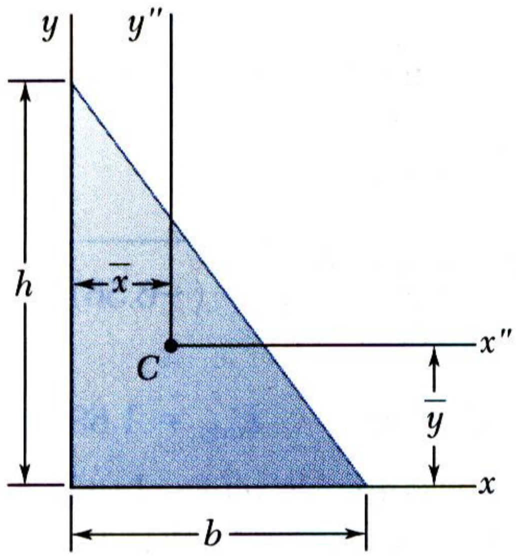
$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el}\bar{y}_{el}dA = \int_0^b x\left(\tfrac{1}{2}\right)h^2\left(1 - \frac{x}{b}\right)^2 dx$$

$$= h^2\int_0^b\left(\frac{x}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2}\right)dx = h^2\left[\frac{x^2}{4} - \frac{x^3}{3b} + \frac{x^4}{8b^2}\right]_0^b$$

$$\boxed{I_{xy} = \tfrac{1}{24}b^2h^2}$$

# Area Moments of Inertia

## Examples




## SOLUTION

- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

$$\bar{x} = \tfrac{1}{3}b \qquad \bar{y} = \tfrac{1}{3}h$$

With the results from part *a*,

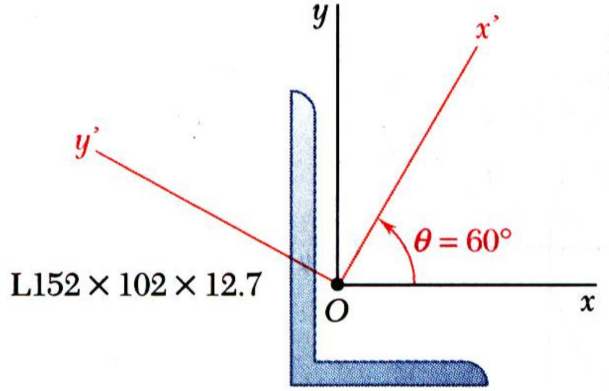
$$I_{xy} = \bar{I}_{x''y''} + \bar{x}\bar{y}A$$

$$\bar{I}_{x''y''} = \tfrac{1}{24}b^2h^2 - \left(\tfrac{1}{3}b\right)\left(\tfrac{1}{3}h\right)\left(\tfrac{1}{2}bh\right)$$

$$\boxed{\bar{I}_{x''y''} = -\tfrac{1}{72}b^2h^2}$$

# Area Moments of Inertia

## Example: Mohr's Circle of Inertia




The moments and product of inertia with respect to the *x* and *y* axes are $I_x$ = 7.24x106 mm$^4$, $I_y$ = 2.61x106 mm$^4$, and $I_{xy} = -2.54\text{x}10^6$ mm$^4$.

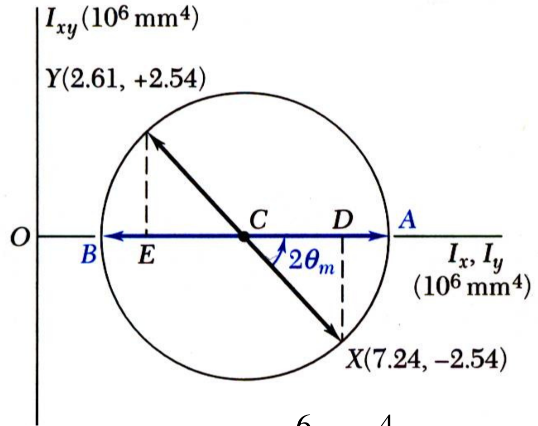
Using Mohr's circle, determine *(a)* the principal axes about *O*, *(b)* the values of the principal moments about *O*, and *(c)* the values of the moments and product of inertia about the *x'* and *y'* axes

<u>SOLUTION</u>:

- Plot the points ($I_x$, $I_{xy}$) and ($I_y$,-$I_{xy}$). Construct Mohr's circle based on the circle diameter between the points.
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.
- Based on the circle, evaluate the moments and product of inertia with respect to the *x'y'* axes.

# Area Moments of Inertia

## Example: Mohr's Circle of Inertia




$$I_x = 7.24\times10^6\,\text{mm}^4$$

$$I_y = 2.61\times10^6\,\text{mm}^4$$

$$I_{xy} = -2.54\times10^6\,\text{mm}^4$$

SOLUTION:

- Plot the points ($I_x$, $I_{xy}$) and ($I_y$,-$I_{xy}$). Construct Mohr's circle based on the circle diameter between the points.

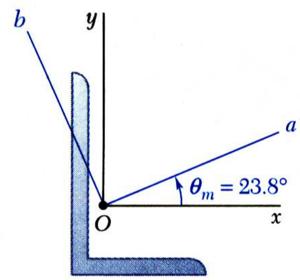
$$OC = I_{ave} = \tfrac{1}{2}\left(I_x + I_y\right) = 4.925\times10^6\,\text{mm}^4$$

$$CD = \tfrac{1}{2}\left(I_x - I_y\right) = 2.315\times10^6\,\text{mm}^4$$

$$R = \sqrt{(CD)^2 + (DX)^2} = 3.437\times10^6\,\text{mm}^4$$

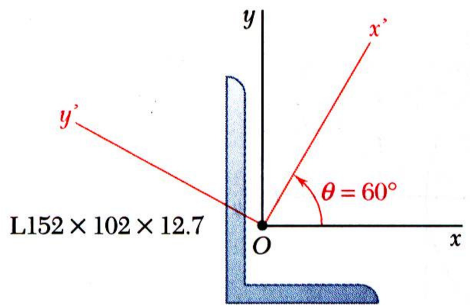
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.

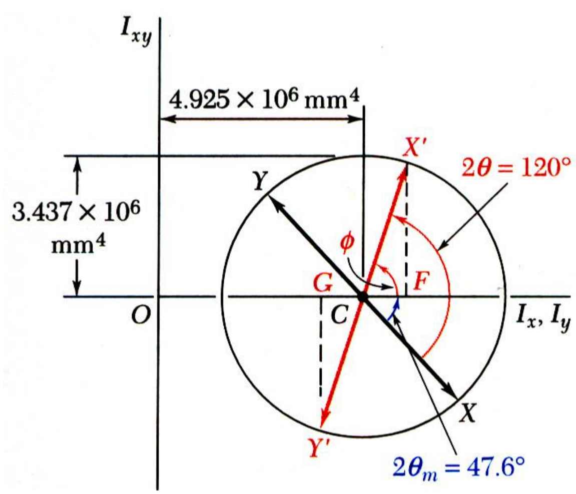
$$\tan 2\theta_m = \frac{DX}{CD} = 1.097 \quad 2\theta_m = 47.6° \qquad \boxed{\theta_m = 23.8°}$$



# Area Moments of Inertia

## Example: Mohr's Circle of Inertia




$$OC = I_{ave} = 4.925 \times 10^6 \,\text{mm}^4$$

$$R = 3.437 \times 10^6 \,\text{mm}^4$$

- Based on the circle, evaluate the moments and product of inertia with respect to the *x'y'* axes.

  The points *X'* and *Y'* corresponding to the *x'* and *y'* axes are obtained by rotating *CX* and *CY* counterclockwise through an angle $\theta = 2(60^o) = 120^o$. The angle that *CX'* forms with the horz is $\phi = 120^o - 47.6^o = 72.4^o$.

$$I_{x'} = OF = OC + CX' \cos\varphi = I_{ave} + R\cos 72.4^o$$

$$\boxed{I_{x'} = 5.96 \times 10^6 \,\text{mm}^4}$$

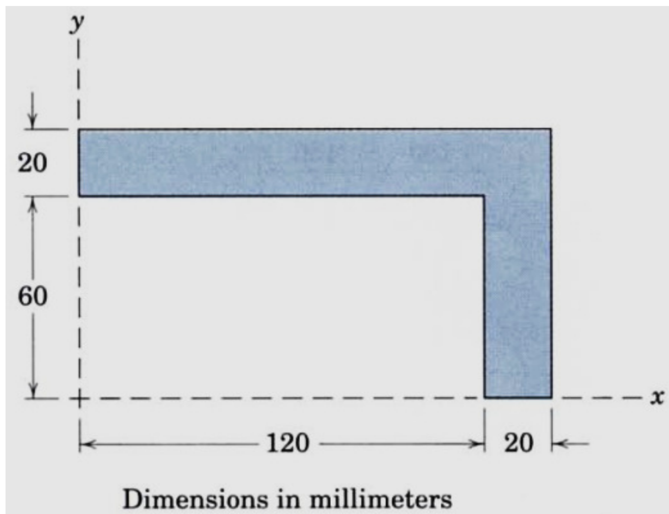
$$I_{y'} = OG = OC - CY' \cos\varphi = I_{ave} - R\cos 72.4^o$$

$$\boxed{I_{y'} = 3.89 \times 10^6 \,\text{mm}^4}$$

$$I_{x'y'} = FX' = CY' \sin\varphi = R\sin 72.4^o$$

$$\boxed{I_{x'y'} = 3.28 \times 10^6 \,\text{mm}^4}$$

Determine the product of the inertia of the shaded area shown below about the *x*-*y* axes.



**Solution**:

Parallel axis theorem: $I_{xy} = \bar{I}_{xy} + d_x\, d_y A$

Both areas (1) and (2) are symmetric @ their centroidal

Axis → $\bar{I}_{xy} = 0$ for both area.

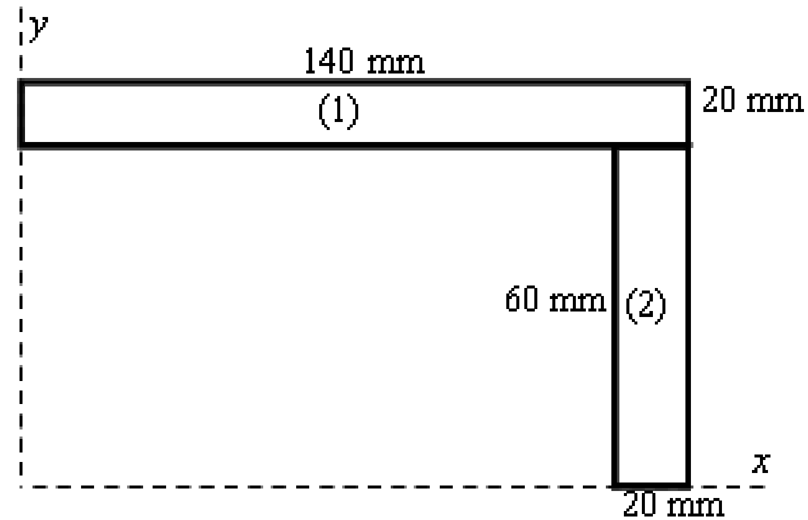
Therefore, for Area (1): $I_{xy1} = d_{x1}\, d_{y1} A_1$

$I_{xy1} = 20\times140\times70\times70 = 13.72\times10^6$ mm $^4$

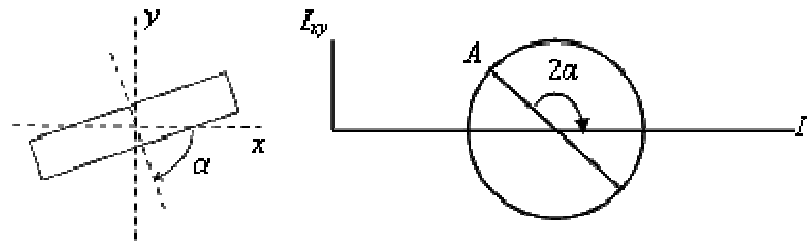
Similarly, for Area (2): $I_{xy2} = d_{x2}\, d_{y2} A_2$

$I_{xy2} = 60\times20\times130\times30 = 4.68\times10^6$ mm $^4$

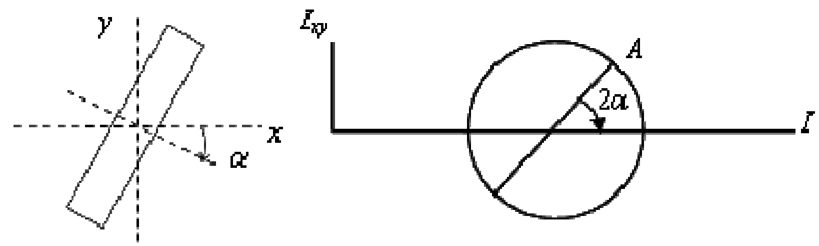
Total $I_{xy} = I_{xy1} + I_{xy2} = 18.40\times10^6$ mm $^4$



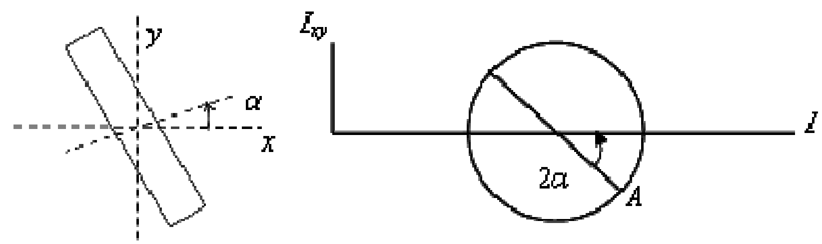
$I_x < I_y\,, \quad I_{xy}\,(+)$



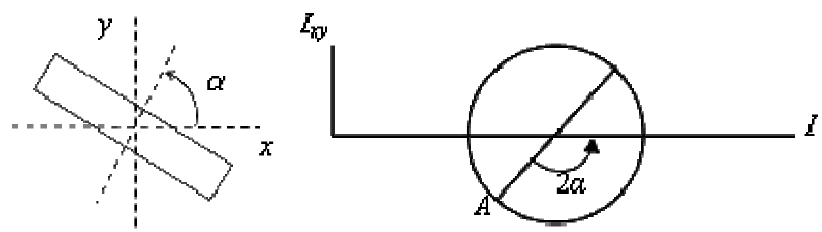

$I_x > I_y\,, \quad I_{xy}\,(+)$




$I_x > I_y\,, \quad I_{xy}\,(-)$



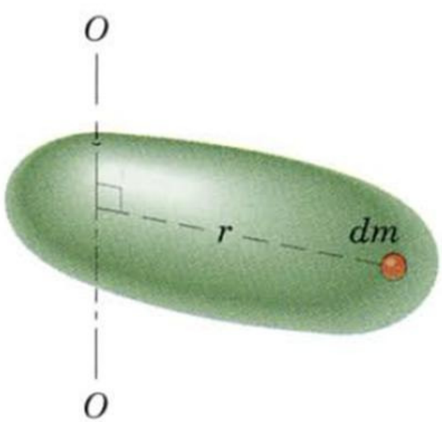

$I_x < I_y\,, \quad I_{xy}\,(-)$



# Area Moments of Inertia

## Mass Moments of Inertia ($I$): Important in Rigid Body Dynamics

- $I$ is a measure of distribution of mass of a rigid body w.r.t. the axis in question (constant property for that axis).
- Units are (mass)(length)$^2$ → kg.m$^2$




Consider a three dimensional body of mass *m*
Mass moment of inertia of this body about axis *O-O*:

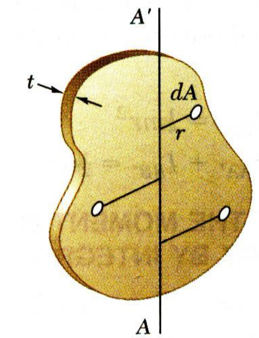
$$I = \int r^2 \, dm$$

Integration is over the entire body.

*r* = perpendicular distance of the mass element *dm* from the axis *O-O*

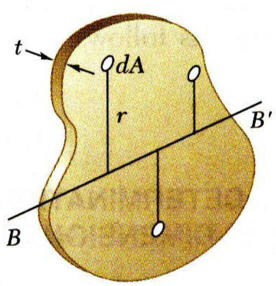
# Area Moments of Inertia

## Moments of Inertia of Thin Plates



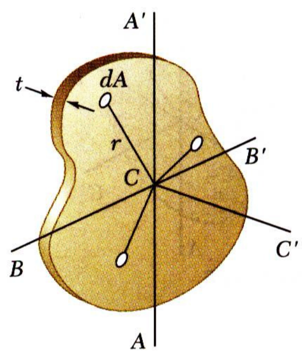

- For a thin plate of uniform thickness $t$ and homogeneous material of density $\rho$, the mass moment of inertia with respect to axis $AA'$ contained in the plate is

$$I_{AA'} = \int r^2 dm = \rho t \int r^2 dA$$
$$= \rho t \, I_{AA',area}$$




- Similarly, for perpendicular axis $BB'$ which is also contained in the plate,

$$I_{BB'} = \rho t \, I_{BB',area}$$

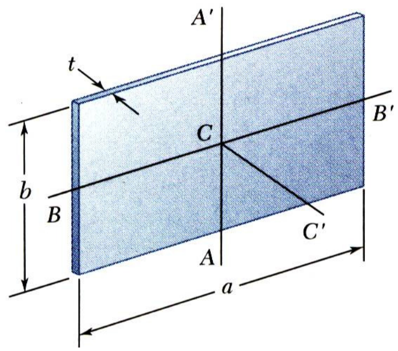



- For the axis $CC'$ which is perpendicular to the plate,

$$I_{CC'} = \rho t \, J_{C,area} = \rho t \left( I_{AA',area} + I_{BB',area} \right)$$
$$= I_{AA'} + I_{BB'}$$

# Area Moments of Inertia

## Moments of Inertia of Thin Plates

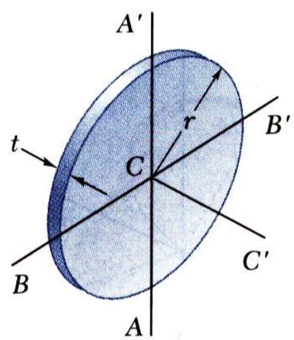



- For the principal centroidal axes on a rectangular plate,

$$I_{AA'} = \rho t \, I_{AA',area} = \rho t \left(\tfrac{1}{12} a^3 b\right) = \tfrac{1}{12} m a^2$$

$$I_{BB'} = \rho t \, I_{BB',area} = \rho t \left(\tfrac{1}{12} a b^3\right) = \tfrac{1}{12} m b^2$$

$$I_{CC'} = I_{AA',mass} + I_{BB',mass} = \tfrac{1}{12} m \left(a^2 + b^2\right)$$

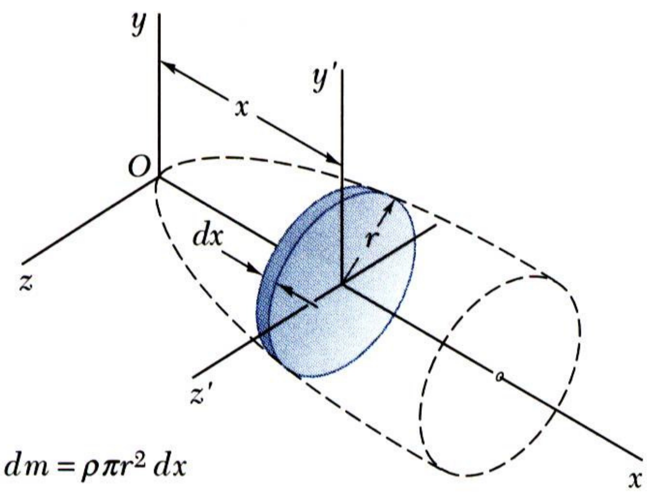



- For centroidal axes on a circular plate,

$$I_{AA'} = I_{BB'} = \rho t \, I_{AA',area} = \rho t \left(\tfrac{1}{4} \pi r^4\right) = \tfrac{1}{4} m r^2$$

# Area Moments of Inertia

## Moments of Inertia of a 3D Body by Integration




$$dm = \rho \pi r^2\, dx$$

$$dI_x = \frac{1}{2} r^2\, dm$$

$$dI_y = dI_{y'} + x^2\, dm = \left(\frac{1}{4} r^2 + x^2\right) dm$$

$$dI_z = dI_{z'} + x^2\, dm = \left(\frac{1}{4} r^2 + x^2\right) dm$$

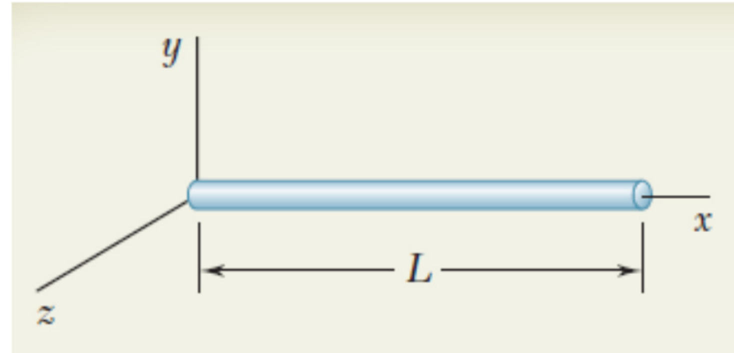
- Moment of inertia of a homogeneous body is obtained from double or triple integrations of the form

$$I = \rho \int r^2 dV$$

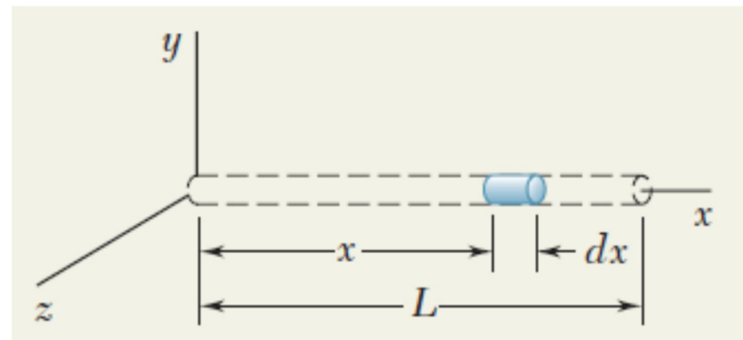
- For bodies with two planes of symmetry, the moment of inertia may be obtained from a single integration by choosing thin slabs perpendicular to the planes of symmetry for *dm*.

- The moment of inertia with respect to a particular axis for a composite body may be obtained by adding the moments of inertia with respect to the same axis of the components.

Q. No. Determine the moment of inertia of a slender rod of length L and mass m with respect to an axis which is perpendicular to the rod and passes through one end of the rod.



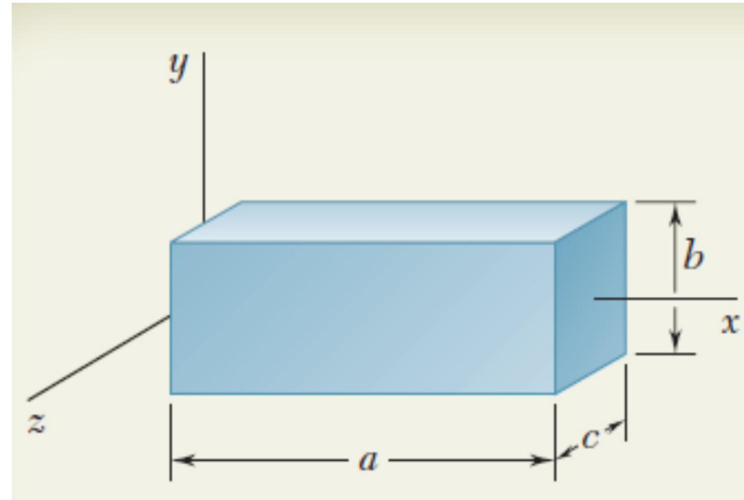

**Solution**




$$dm = \frac{m}{L}dx$$

$$I_y = \int x^2\,dm = \int_0^L x^2\frac{m}{L}dx = \left[\frac{m}{L}\frac{x^3}{3}\right]_0^L \qquad I_y = \tfrac{1}{3}mL^2$$

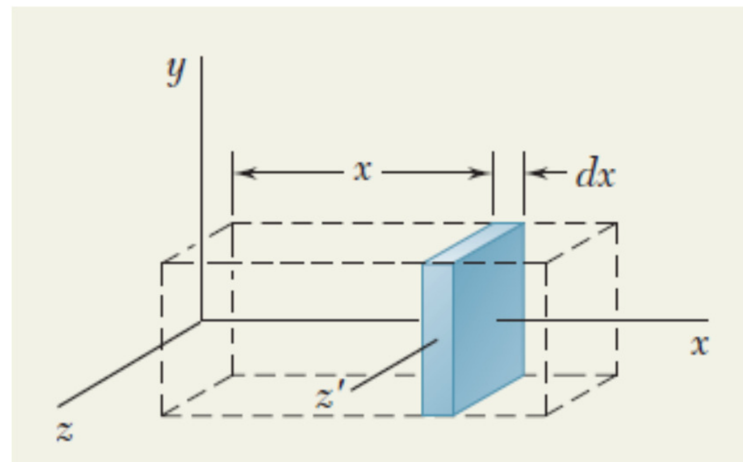
Q. No. For the homogeneous rectangular prism shown, determine the moment of inertia with respect to z axis




**Solution**

$$dm = \rho bc\, dx \qquad dI_{z'} = \tfrac{1}{12} b^2\, dm$$

$$dI_z = dI_{z'} + x^2\, dm = \tfrac{1}{12} b^2\, dm + x^2\, dm = (\tfrac{1}{12} b^2 + x^2)\rho bc\, dx$$

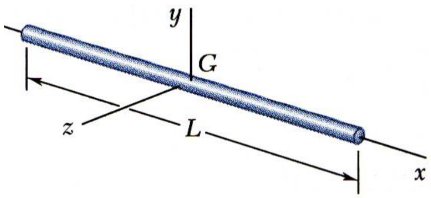



$$I_z = \int dI_z = \int_0^a (\tfrac{1}{12} b^2 + x^2)\rho bc\, dx = \rho abc(\tfrac{1}{12} b^2 + \tfrac{1}{3} a^2)$$

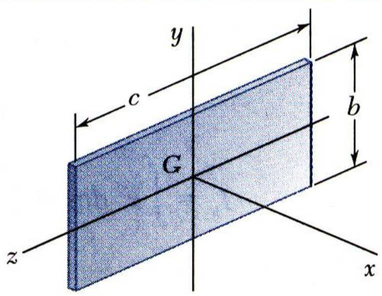
$$I_z = m(\tfrac{1}{12} b^2 + \tfrac{1}{3} a^2) \qquad I_z = \tfrac{1}{12} m(4a^2 + b^2)$$

# Area Moments of Inertia

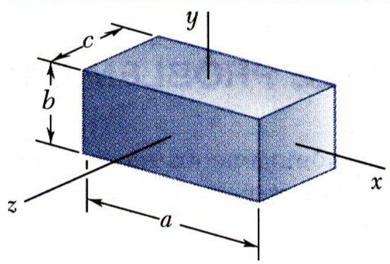
MI of some common geometric shapes



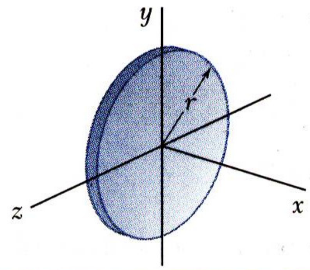

$$I_y = I_z = \frac{1}{12} mL^2$$



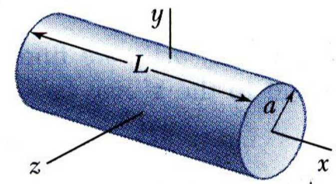

$$I_x = \frac{1}{12} m(b^2 + c^2)$$
$$I_y = \frac{1}{12} mc^2$$
$$I_z = \frac{1}{12} mb^2$$



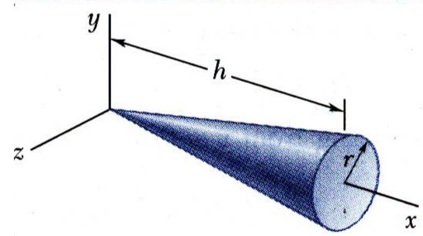

$$I_x = \frac{1}{12} m(b^2 + c^2)$$
$$I_y = \frac{1}{12} m(c^2 + a^2)$$
$$I_z = \frac{1}{12} m(a^2 + b^2)$$



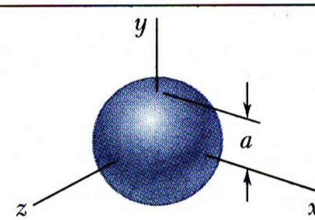

$$I_x = \frac{1}{2} mr^2$$
$$I_y = I_z = \frac{1}{4} mr^2$$




$$I_x = \frac{1}{2} ma^2$$
$$I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$$

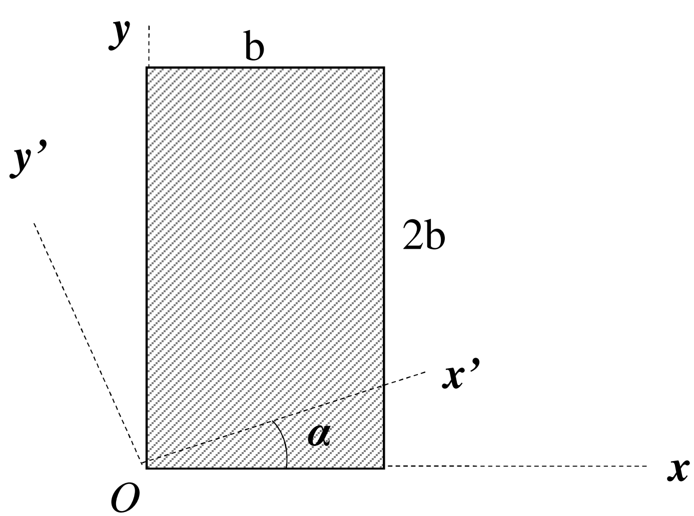



$$I_x = \frac{3}{10} ma^2$$
$$I_y = I_z = \frac{3}{5} m(\frac{1}{4} a^2 + h^2)$$




$$I_x = I_y = I_z = \frac{2}{5} ma^2$$

Q. Determine the angle $\alpha$ which locates the principal axes of inertia through point $O$ for the rectangular area (Figure 5). Construct the Mohr's circle of inertia and specify the corresponding values of $I_{max}$ and $I_{min}$.



$I_x = \frac{1}{3}b(2b)^3 = \frac{8}{3}b^4$

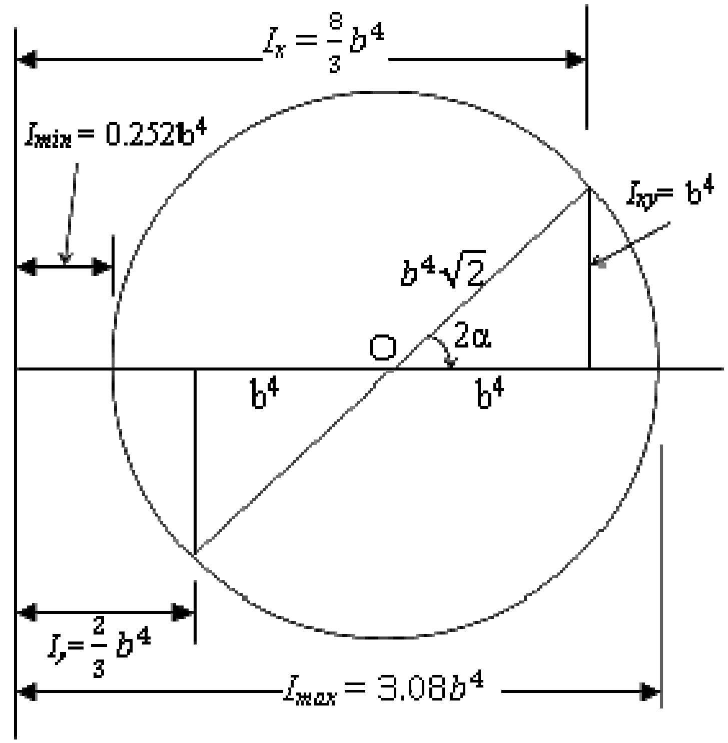
$I_y = \frac{1}{3}2b(b)^3 = \frac{2}{3}b^4$

$I_{xy} = 2b^2(\frac{b}{2})(b) = b^4$

With this data, plot the Mohr's circle, and using trigonometry calculate the angle $2\alpha$

$2\alpha = tan^{-1}(b^4/b^4) = 45^o$

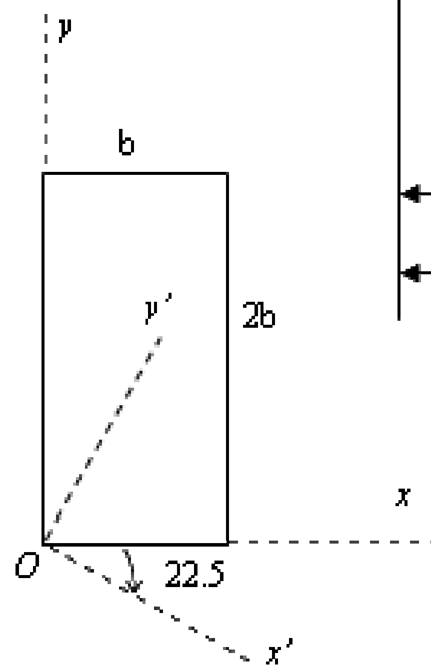
Therefore, $\alpha = 22.5^o$ (clockwise w.r.t. $x$)




From Mohr's Circle:

$I_{max} = \frac{5}{3}b^4 + b^4\sqrt{2} = 3.08b^4$

$I_{min} = \frac{5}{3}b^4 - b^4\sqrt{2} = 0.252b^4$

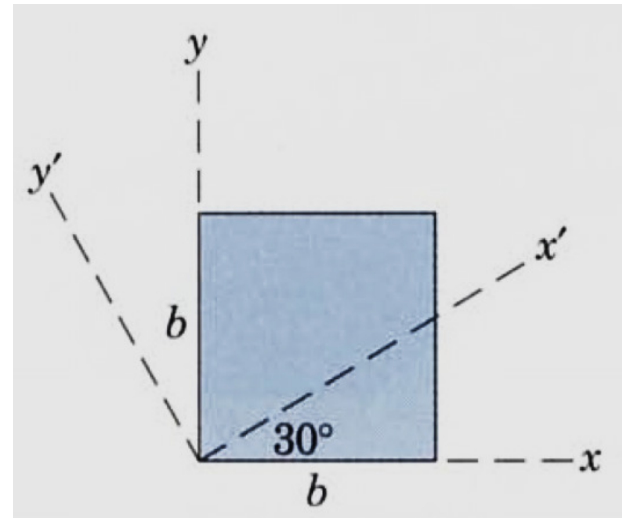


Q. No. Determine the moments and product of inertia of the area of the square with respect to the $x'$- $y'$ axes.

$I_x = \frac{1}{3}b(b^3) = \frac{1}{3}b^4$; $\quad I_y = \frac{1}{3}b^4$

$I_{xy} = 0 + (\frac{b}{2} \times \frac{b}{2} \times b^2) = \frac{1}{4}b^4$

With $\theta = 30^o$, using the equation of moment of inertia about any inclined axes




$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} Cos\ 2\theta - I_{xy}\ Sin\ 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} Cos\ 2\theta + I_{xy}\ Sin\ 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2} Sin\ 2\theta + I_{xy}\ Cos\ 2\theta$$

we get,

$\rightarrow I_{x'} = \frac{b^4}{3} + 0 - \frac{1}{4}b^4 Sin\ 60^\circ = (\frac{1}{3} - \frac{\sqrt{3}}{8})b^4 = 0.1168b^4$

$\rightarrow I_{y'} = \frac{b^4}{3} + 0 + \frac{1}{4}b^4 Sin\ 60^\circ = (\frac{1}{3} + \frac{\sqrt{3}}{8})b^4 = 0.5498b^4$

$\rightarrow I_{xy'} = 0 + (\frac{1}{4}b^4 \times \frac{1}{2}) = (\frac{1}{8} \times b^4) = 0.1250b^4$

Alternatively, Mohr's Circle may be used to determine the three quantities.