ME 101: Engineering Mechanics

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Virtual Work

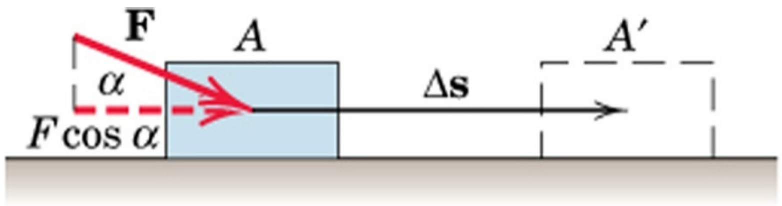
Method of Virtual Work

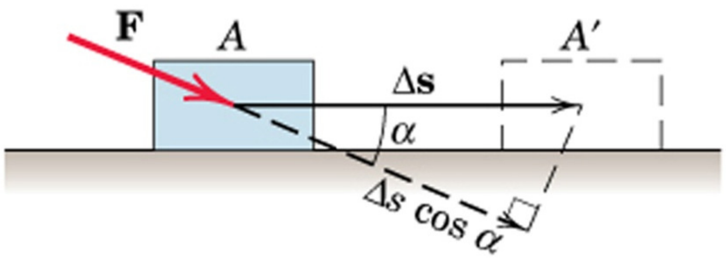
- Previous methods (FBD, $\sum F$, $\sum M$) are generally employed for a body whose equilibrium position is known or specified
- For problems in which bodies are composed of interconnected members that can move relative to each other.
- → various equilibrium configurations are possible and must be examined.
- → previous methods can still be used but are not the direct and convenient.
- **Method of Virtual Work** is suitable for analysis of multi-link structures (pin-jointed members) which change configuration
- effective when a simple relation can be found among the disp. of the pts of application of various forces involved
- → based on the concept of work done by a force
- → enables us to examine stability of systems in equilibrium

Scissor Lift Platform

Virtual Work

Work done by a Force (U)

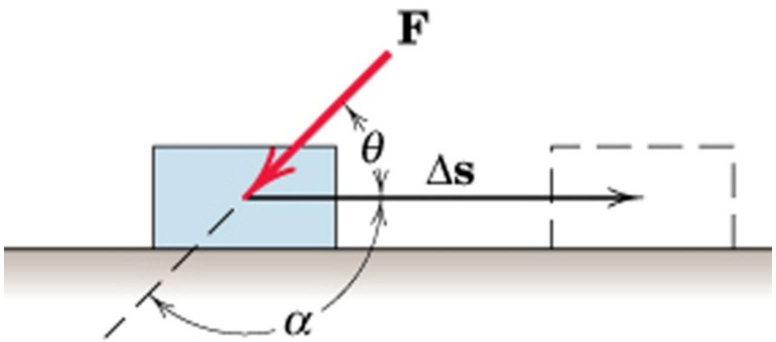




U = work done by the component of the force in the direction of the displacement times the displacement

$$U = (F \cos \alpha)\, \Delta s$$ or $$U = F(\Delta s \cos \alpha)$$

Since same results are obtained irrespective of the direction in which we resolve the vectors → Work is a scalar quantity



+U → Force and Disp in same direction

- U → Force and Disp in opposite direction

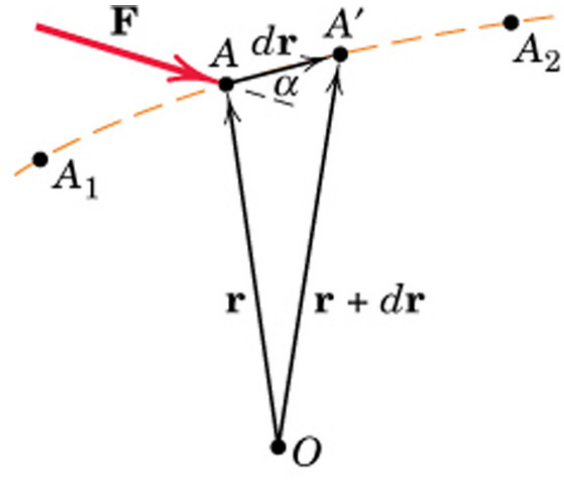
$$U = (F \cos \alpha)\, \Delta s = -(F \cos \theta)\, \Delta s$$

Virtual Work

Work done by a Force (U)

Generalized Definition of Work

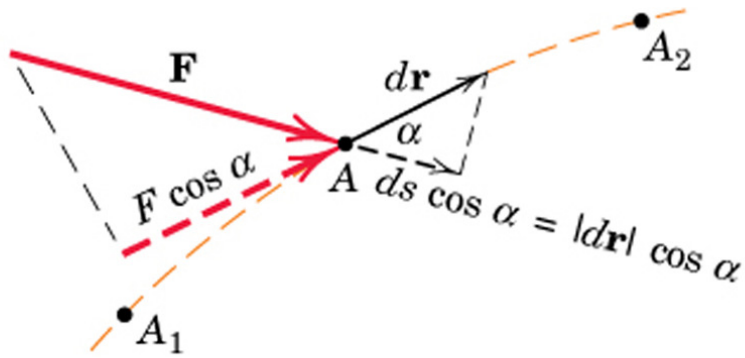
Work done by **F** during displacement $d\mathbf{r}$



$$\boxed{dU = \mathbf{F}\cdot d\mathbf{r}}$$

→ $dU = F\, ds \cos\alpha$

Expressing **F** and $d\mathbf{r}$ in terms of their rectangular components



$$\begin{aligned} dU &= (\mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z)\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz) \\ &= F_x\,dx + F_y\,dy + F_z\,dz \end{aligned}$$

Total work done by **F** from A_1 to A_2

$$U = \int \mathbf{F}\cdot d\mathbf{r} = \int (F_x\,dx + F_y\,dy + F_z\,dz)$$ → $$U = \int F\cos\alpha\,ds$$

Virtual Work

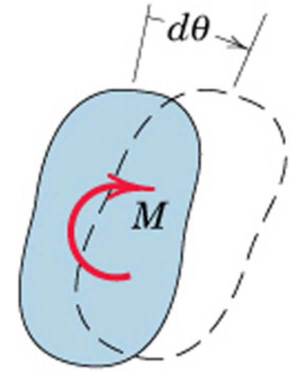
Work done by a Couple (U)

Small rotation of a rigid body:

- translation to $A'B'$

  → work done by F during disp AA' will be equal and opposite to work done by $-F$ during disp BB'

  → total work done is zero



- rotation of A' about B' to A''

→ work done by F during disp AA'' :

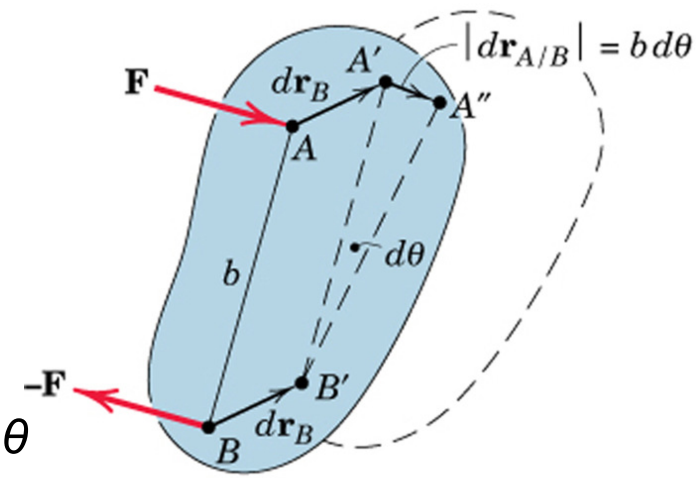
$U = \mathbf{F}.d\mathbf{r}_{A/B} = Fbd\theta$

Since $M = Fb$

→ $\boxed{dU = M\,d\theta}$

$+M$ → M has same sense as θ

$-M$ → M has opp sense as θ



Total word done by a couple during a finite rotation in its plane:

$$U = \int M\,d\theta$$

Virtual Work

Dimensions and Units of Work

(Force) x (Distance) → Joule (J) = N.m

→ Work done by a force of 1 Newton moving through a distance of 1 m in the direction of the force

→ Dimensions of Work of a Force and Moment of a Force are same though they are entirely different physical quantities.

→ **Work** is a scalar given by dot product; involves product of a force and distance, both measured along the same line

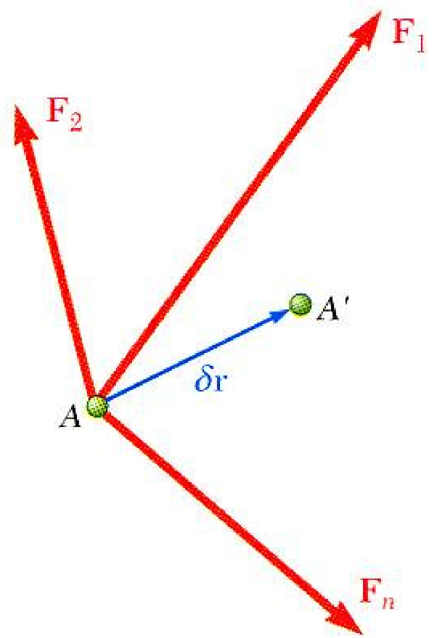
→ **Moment** is a vector given by the cross product; involves product of a force and distance measured at right angles to the force

→ Units of Work: Joule

→ Units of Moment: N.m

Virtual Work

Virtual Work: Disp does not really exist but only is assumed to exist so that we may compare various possible equilibrium positions to determine the correct one.



- *Imagine* the small *virtual displacement* of particle which is acted upon by several forces.
- The corresponding *virtual work*,

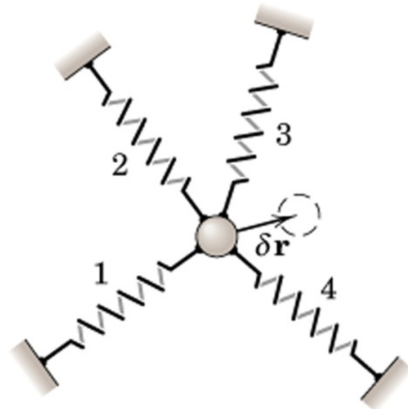
$$\delta U = \vec{F}_1 \cdot \delta\vec{r} + \vec{F}_2 \cdot \delta\vec{r} + \vec{F}_3 \cdot \delta\vec{r} = \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3\right) \cdot \delta\vec{r}$$
$$= \vec{R} \cdot \delta\vec{r}$$

Principle of Virtual Work:

- If a particle is in equilibrium, the total virtual work of forces acting on the particle is zero for any virtual displacement.
- If a rigid body is in equilibrium, the total virtual work of external forces acting on the body is zero for any virtual displacement of the body.
- If a system of connected rigid bodies remains connected during the virtual displacement, only the work of the external forces need be considered since work done by internal forces (equal, opposite, and collinear) cancels each other.

Virtual Work

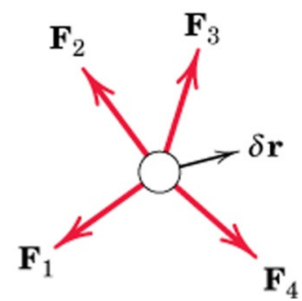
Equilibrium of a Particle



Total virtual work done on the particle due to virtual displacement $\delta\mathbf{r}$:

$$\delta U = \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \mathbf{F}_3 \cdot \delta\mathbf{r} + \cdots = \Sigma\mathbf{F} \cdot \delta\mathbf{r}$$

Expressing $\sum\mathbf{F}$ in terms of scalar sums and $\delta\mathbf{r}$ in terms of its component virtual displacements in the coordinate directions:



$$\begin{aligned}\delta U &= \Sigma\mathbf{F} \cdot \delta\mathbf{r} = (\mathbf{i}\,\Sigma F_x + \mathbf{j}\,\Sigma F_y + \mathbf{k}\,\Sigma F_z) \cdot (\mathbf{i}\,\delta x + \mathbf{j}\,\delta y + \mathbf{k}\,\delta z) \\ &= \Sigma F_x\,\delta x + \Sigma F_y\,\delta y + \Sigma F_z\,\delta z = 0\end{aligned}$$

The sum is zero since $\sum\mathbf{F} = 0$, which gives $\sum F_x = 0$, $\sum F_y = 0$, $\sum F_z = 0$

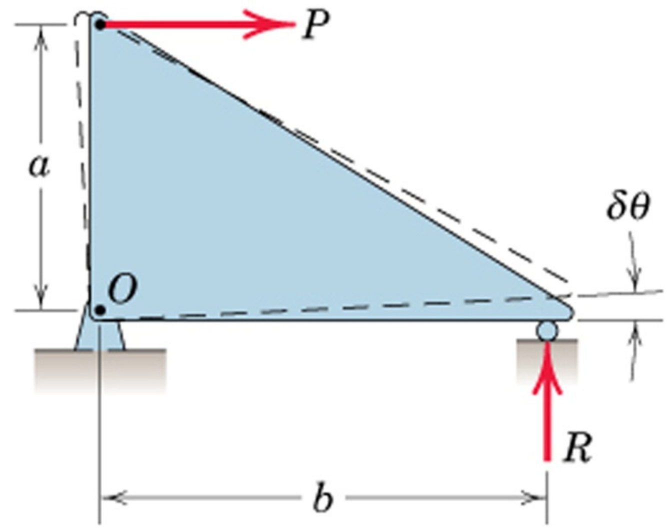
Alternative Statement of the equilibrium: $\delta U = 0$

This condition of zero virtual work for equilibrium is both necessary and sufficient since we can apply it to the three mutually perpendicular directions

→ **3 conditions of equilibrium**

Virtual Work: Applications of Principle of Virtual Work

Equilibrium of a Rigid Body



Total virtual work done on the entire rigid body is zero since virtual work done on each Particle of the body in equilibrium is zero.

Weight of the body is negligible.

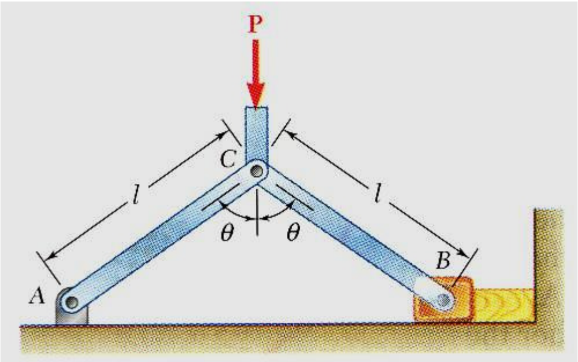
Work done by P = $-Pa\,\delta\theta$
Work done by R = $+Rb\,\delta\theta$

Principle of Virtual Work: $\delta U = 0$:
$-Pa\,\delta\theta + Rb\,\delta\theta = 0$

→ $Pa - Rb = 0$
→ Equation of Moment equilibrium @ O.
→ Nothing gained by using the Principle of Virtual Work for a single rigid body

Virtual Work: Applications of Principle of Virtual Work

Determine the force exerted by the vice on the block when a given force P is applied at C. Assume that there is no friction.



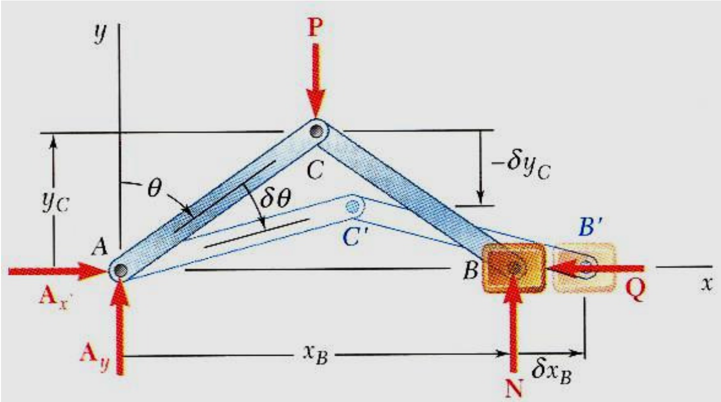
- Consider the work done by the external forces for a virtual displacement $\delta\theta$. $\delta\theta$ is a positive increment to θ in bottom figure. Only the forces P and Q produce nonzero work.
- x_B increases while y_C decreases

→ +ve increment for x_B: δx_B → $\delta U_Q = -Q\delta x_B$ (opp. Sense)

→ -ve increment for y_C: $-\delta y_C$ → $\delta U_P = +P(-\delta y_C)$ (same Sense)

$$\delta U = 0 = \delta U_Q + \delta U_P = -Q\,\delta x_B - P\,\delta y_C$$

Expressing x_B and y_C in terms of θ and differentiating w.r.t. θ



$$x_B = 2l\sin\theta \qquad y_C = l\cos\theta$$

$$\delta x_B = 2l\cos\theta\,\delta\theta \qquad \delta y_C = -l\sin\theta\,\delta\theta$$

$$0 = -2Ql\cos\theta\,\delta\theta + Pl\sin\theta\,\delta\theta$$

$$Q = \tfrac{1}{2}P\tan\theta$$

By using the method of virtual work, all unknown reactions were eliminated. $\sum M_A$ would eliminate only two reactions.

- If the virtual displacement is consistent with the constraints imposed by supports and connections, only the work of loads, applied forces, and friction forces need be considered.

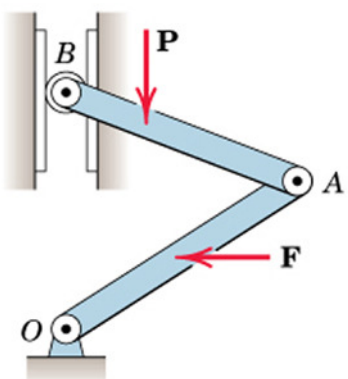
Virtual Work

Principle of Virtual Work

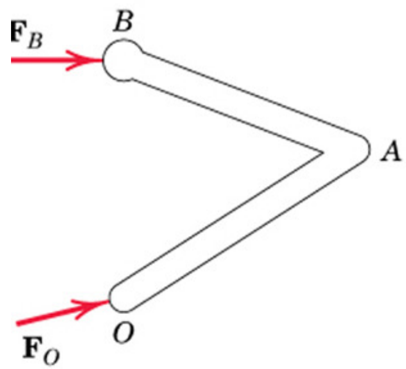
Virtual Work done by external active forces on an ideal mechanical system in equilibrium is zero for any and all virtual displacements consistent with the constraints

$$\delta U = 0$$

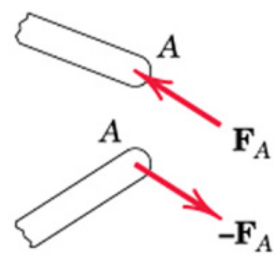
Three types of forces act on interconnected systems made of rigid members



Active Forces: Work Done
Active Force Diagram



Reactive Forces
No Work Done



Internal Forces
No Work Done

Virtual Work

Major Advantages of the Virtual Work Method

- It is not necessary to dismember the systems in order to establish relations between the active forces.
- Relations between active forces can be determined directly without reference to the reactive forces.

→ The method is particularly useful in determining the position of equilibrium of a system under known loads (This is in contrast to determining the forces acting on a body whose equilibrium position is known – studied earlier).

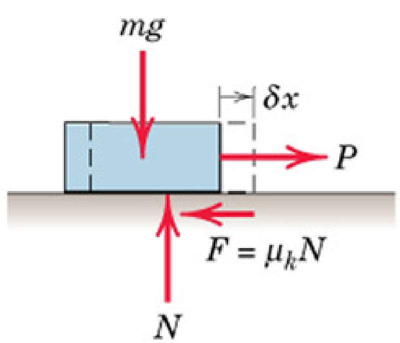
→ The method requires that internal frictional forces do negligible work during any virtual displacement.

→ If internal friction is appreciable, work done by internal frictional forces must be included in the analysis.

Virtual Work

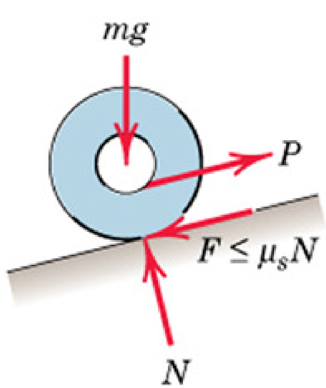
Systems with Friction

- So far, the Principle of virtual work was discussed for “ideal” systems.
- If significant friction is present in the system (“Real” systems), work done by the external active forces (input work) will be opposed by the work done by the friction forces.



During a virtual displacement δx:

Work done by the kinetic friction force is: $-\mu_k N \delta x$



During rolling of a wheel:
the static friction force does no work if the wheel does not slip as it rolls.

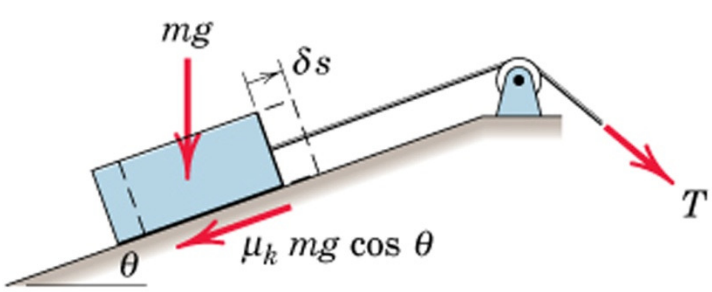
Virtual Work

Mechanical Efficiency (e)

- Output work of a machine is always less than the input work because of energy loss due to friction.

$$e = \frac{\text{Output Work}}{\text{Input Work}}$$

For simple machines with SDOF & which operates in uniform manner, mechanical efficiency may be determined using the method of Virtual Work



For the virtual displacement δs: Output Work is that necessary to elevate the block = $mg\ \delta s\ \sin\theta$

Input Work = $T\ \delta s = mg\ \sin\theta\ \delta s + \mu_k\ mg\ \cos\theta\ \delta s$

The efficiency of the inclined plane is:

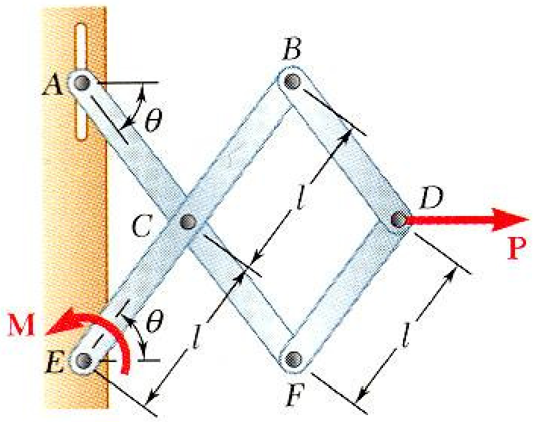
$$e = \frac{mg\,\delta s \sin\theta}{mg(\sin\theta + \mu_k \cos\theta)\delta s} = \frac{1}{1 + \mu_k \cot\theta}$$

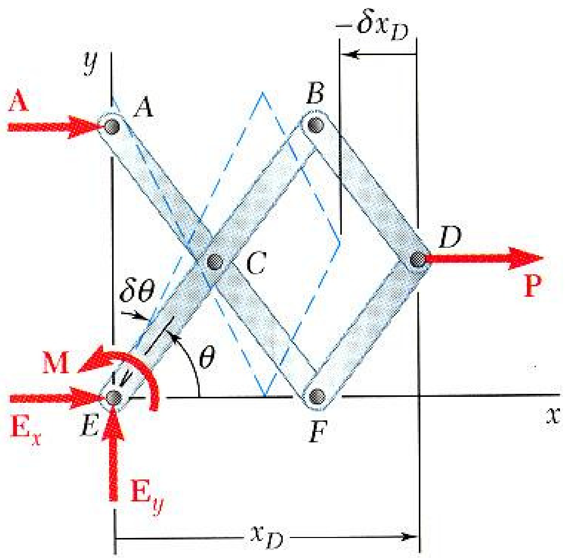
As friction decreases, Efficiency approaches unity

Virtual Work

Example

Determine the magnitude of the couple *M* required to maintain the equilibrium of the mechanism.





SOLUTION:

- Apply the principle of virtual work

$$\delta U = 0 = \delta U_M + \delta U_P$$
$$0 = M\delta\theta + P\delta x_D$$

$$x_D = 3l\cos\theta$$
$$\delta x_D = -3l\sin\theta\delta\theta$$

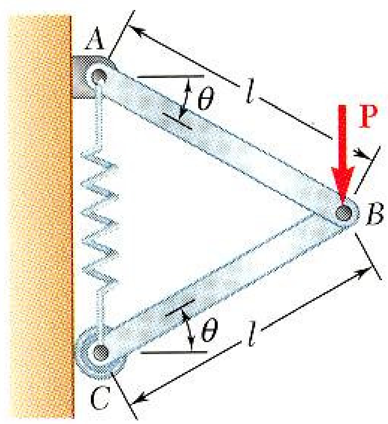
$$0 = M\delta\theta + P(-3l\sin\theta\delta\theta)$$

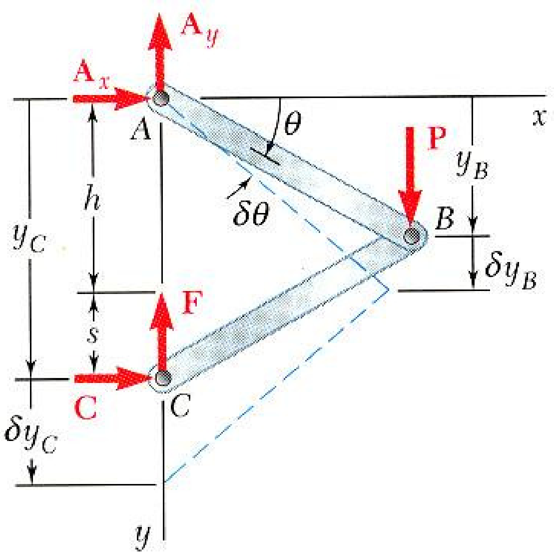
$$\boxed{M = 3Pl\sin\theta}$$

Virtual Work

Example

Determine the expressions for θ and the tension in the spring which correspond to the equilibrium position of the spring. The unstretched length of the spring is *h* and the constant of the spring is k. Neglect the weight of the mechanism.





SOLUTION:

- Apply the principle of virtual work

$$\delta U = \delta U_B + \delta U_F = 0$$

$$0 = P\,\delta y_B - F\,\delta y_C$$

$$y_B = l\sin\theta \qquad y_C = 2l\sin\theta \qquad F = ks$$

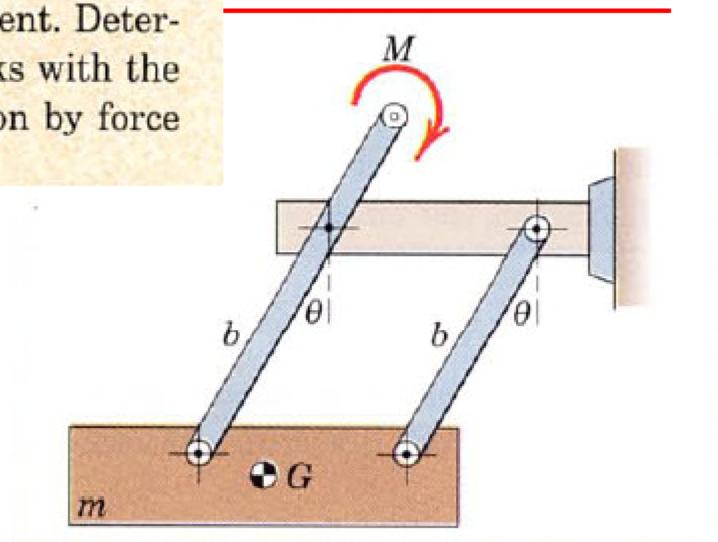
$$\delta y_B = l\cos\theta\,\delta\theta \qquad \delta y_C = 2l\cos\theta\,\delta\theta \qquad = k(y_C - h)$$

$$= k(2l\sin\theta - h)$$

$$0 = P(l\cos\theta\,\delta\theta) - k(2l\sin\theta - h)(2l\cos\theta\,\delta\theta)$$

$$\boxed{\begin{gathered}\sin\theta = \frac{P + 2kh}{4kl} \\ F = \tfrac{1}{2}P\end{gathered}}$$

The mass m is brought to an equilibrium position by the application of the couple M to the end of one of the two parallel links which are hinged as shown. The links have negligible mass, and all friction is assumed to be absent. Determine the expression for the equilibrium angle θ assumed by the links with the vertical for a given value of M. Consider the alternative of a solution by force and moment equilibrium.



$$h = b \cos \theta + c.$$

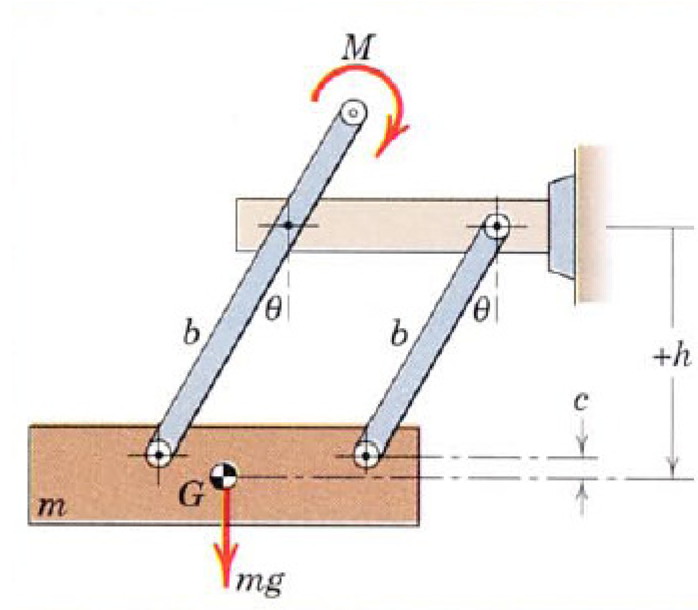
$$\begin{aligned} +mg\,\delta h &= mg\,\delta(b \cos \theta + c) \\ &= mg(-b \sin \theta\,\delta\theta + 0) \\ &= -mgb \sin \theta\,\delta\theta \end{aligned}$$

$[\delta U = 0]$ $\qquad M\,\delta\theta + mg\,\delta h = 0$

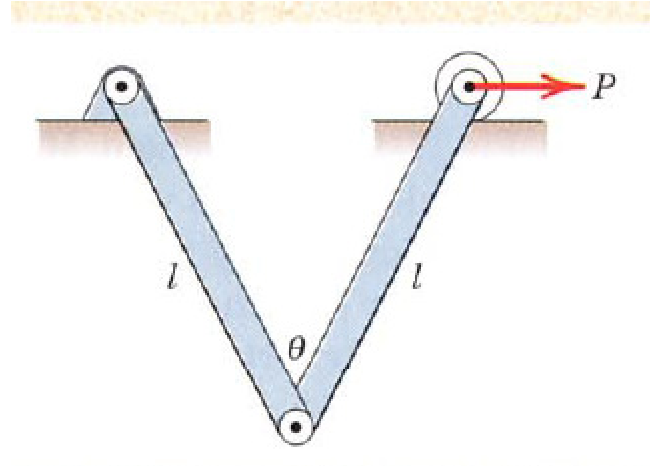
which yields

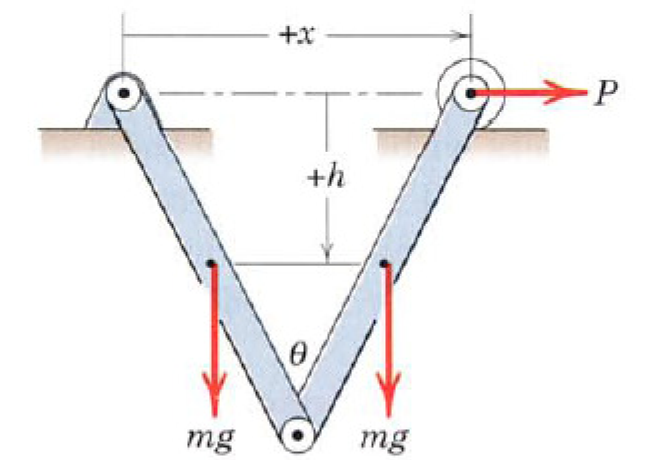
$$M\,\delta\theta = mgb \sin \theta\,\delta\theta$$

$$\theta = \sin^{-1} \frac{M}{mgb}$$



Each of the two uniform hinged bars has a mass m and a length l, and is supported and loaded as shown. For a given force P determine the angle θ for equilibrium.





$[\delta U = 0]$ $\qquad P\,\delta x + 2mg\,\delta h = 0$

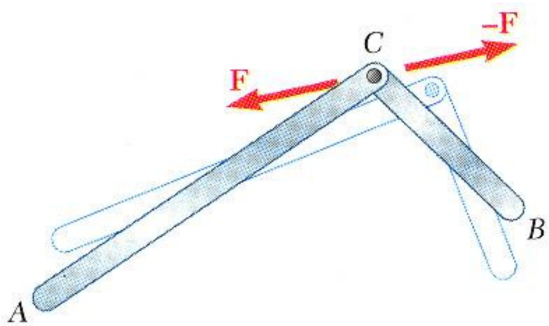
$$x = 2l\sin\frac{\theta}{2} \quad \text{and} \quad \delta x = l\cos\frac{\theta}{2}\,\delta\theta$$

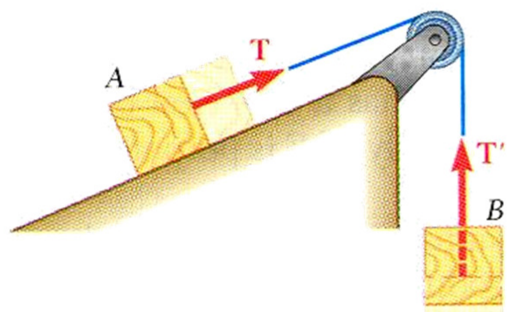
$$h = \frac{l}{2}\cos\frac{\theta}{2} \quad \text{and} \quad \delta h = -\frac{l}{4}\sin\frac{\theta}{2}\,\delta\theta$$

$$Pl\cos\frac{\theta}{2}\,\delta\theta - 2mg\frac{l}{4}\sin\frac{\theta}{2}\,\delta\theta = 0$$

$$\tan\frac{\theta}{2} = \frac{2P}{mg} \quad \text{or} \quad \theta = 2\tan^{-1}\frac{2P}{mg}$$

Virtual Work: Work done by a Force





(b)

Forces which do no work:

- forces applied to fixed points ($ds = 0$)
- forces acting in a dirn normal to the disp ($\cos\alpha = 0$)
- reaction at a frictionless pin due to rotation of a body around the pin
- reaction at a frictionless surface due to motion of a body along the surface
- weight of a body with cg moving horizontally
- friction force on a wheel moving without slipping

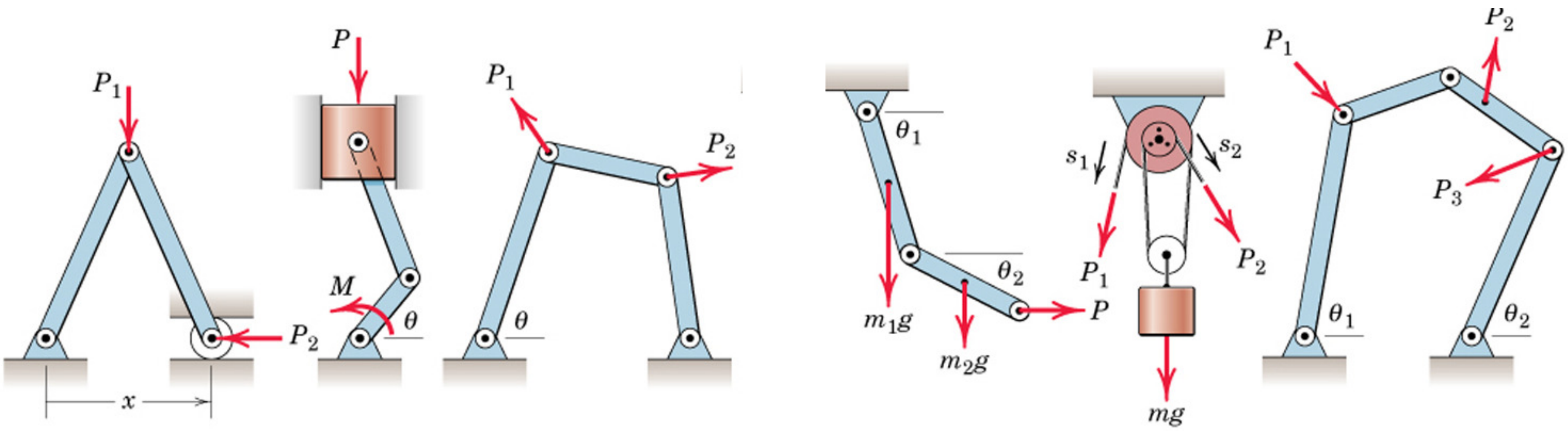
Sum of work done by several forces may be zero:

- bodies connected by a frictionless pin
- bodies connected by an inextensible cord
- internal forces holding together parts of a rigid body

Virtual Work

Degrees of Freedom (DOF)

- Number of independent coordinates needed to specify completely the configuration of system



(*a*) Examples of one-degree-of-freedom systems

(*b*) Examples of two-degree-of-freedom systems

Only one coordinate (displacement or rotation) is needed to establish position of every part of the system

Two independent coordinates are needed to establish position of every part of the system

δU=0 can be applied to each DOF at a time keeping other DOF constant.

ME101 → only SDOF systems

Virtual Work

Potential Energy and Stability

- Till now equilibrium configurations of mechanical systems composed of rigid members was considered for analysis using method of virtual work.

- Extending the method of virtual work to account for mechanical systems which include elastic elements in the form of springs (non-rigid elements).

- Introducing the concept of Potential Energy, which will be used for determining the stability of equilibrium.

- Work done on an elastic member is stored in the member in the form of Elastic Potential Energy V_e.
    - This energy is potentially available to do work on some other body during the relief of its compression or extension.

Virtual Work

Elastic Potential Energy (V_e)

Consider a linear and elastic spring compressed by a force F → $F = kx$

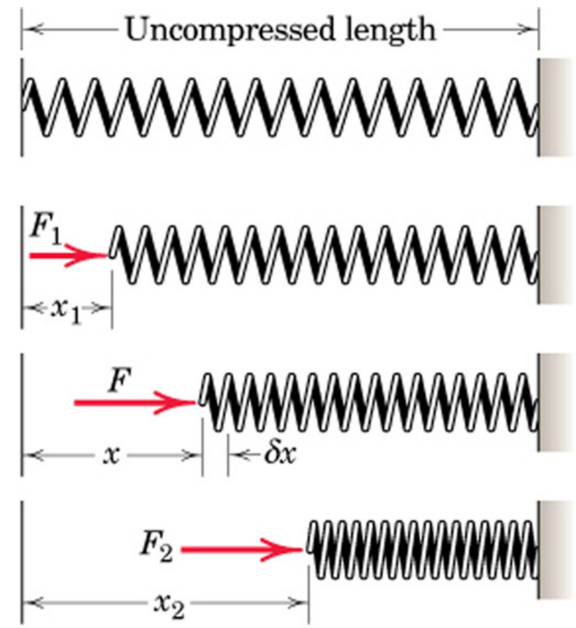
k = spring constant or stiffness of the spring

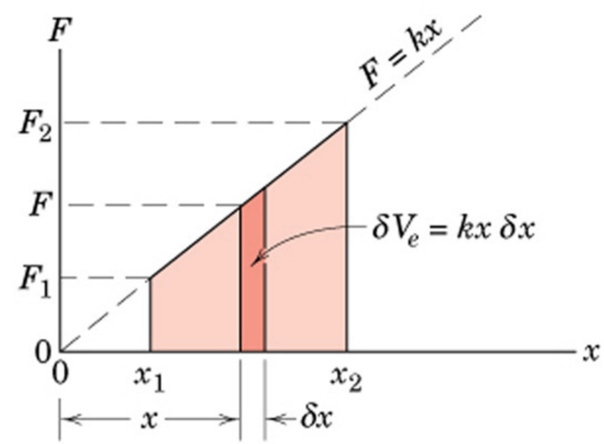
Work done on the spring by F during a movement dx: $dU = F\,dx$

Elastic potential energy of the spring for compression x = total work done on the spring

$$V_e = \int_0^x F\,dx = \int_0^x kx\,dx \qquad \boxed{V_e = \tfrac{1}{2}kx^2}$$

→ Potential Energy of the spring = area of the triangle in the diagram of F versus x from 0 to x





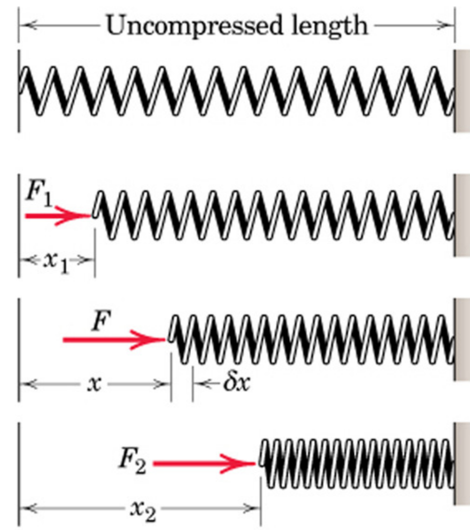
Virtual Work

Elastic Potential Energy

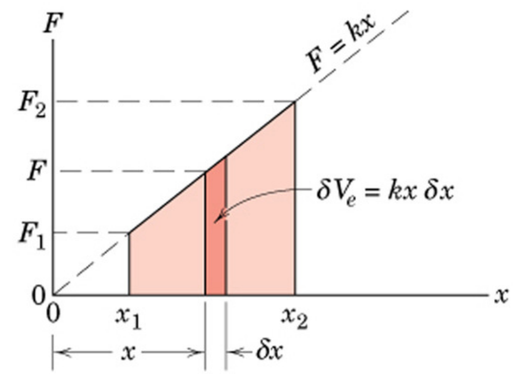
During increase in compression from x_1 to x_2:
Work done on the springs = change in V_e

$$\Delta V_e = \int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}k(x_2{}^2 - x_1{}^2)$$

→ Area of trapezoid from x_1 to x_2



During a virtual displacement δx of the spring:
virtual work done on the spring = virtual change in elastic potential energy



$$\delta V_e = F\,\delta x = kx\,\delta x$$

During the decrease in compression of the spring as it is relaxed from $x=x_2$ to $x=x_1$, the change (final minus initial) in the potential energy of the spring is negative
→ If δx is negative, δV_e is negative

When the spring is being stretched, the force acts in the direction of the displacement → Positive Work on the spring → Increase in the Potential Energy

Virtual Work

Elastic Potential Energy

Work done by the linear spring on the body to which the spring is attached (during displacement of the spring) is the negative of the change in the elastic potential energy of the spring (due to equilibrium).

Torsional Spring: resists the rotation

K = Torsional Stiffness (torque per radian of twist)

θ = angle of twist in radians

Resisting torque, $M = K\theta$

The Potential Energy: $V_e = \int_0^\theta K\theta \, d\theta$ → $\boxed{V_e = \frac{1}{2}K\theta^2}$

This is similar to the expression for the linear extension spring

Units of Elastic Potential Energy → Joules (J)

(same as those of Work)

Virtual Work

Gravitational Potential Energy (V_g)

For an upward displacement δh of the body, work done by the weight ($W=mg$) is:

$\delta U = -\, mg\delta h$

For downward displacement (with h measured positive downward): $\delta U = \; mg\delta h$

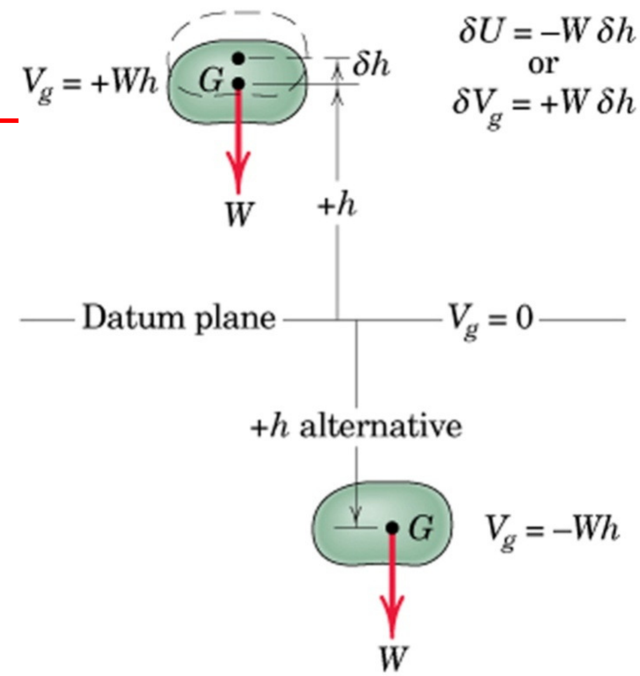
The Gravitational Potential Energy of a body is defined as the work done on the body by a force **equal and opposite to the weight** in bringing the body to the position under consideration from some arbitrary datum plane where the potential energy is defined to be zero.

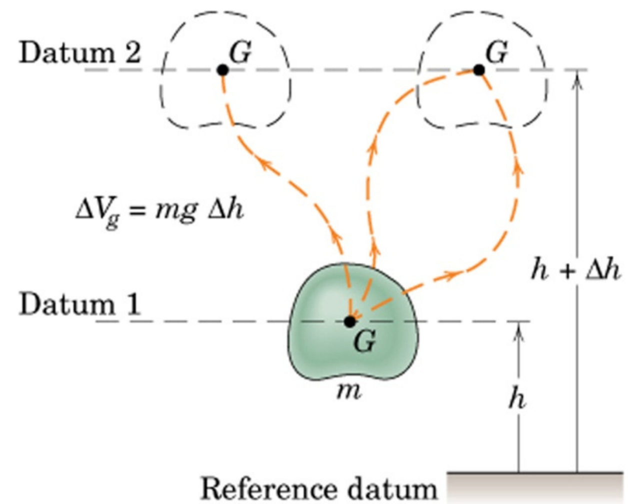
→ V_g is negative of the work done by the weight.

V_g =0 at h=0

→ at height h above the datum plane, $V_g = mgh$

→ at height h below the datum plane, $V_g = -mgh$





Virtual Work

Energy Equation

Work done by the linear spring on the body to which the spring is attached (during displacement of the spring) is the negative of the change in the elastic potential energy of the spring.

Work done by the gravitational force or weight mg is the negative of the change in gravitational potential energy

Virtual Work equation to a system with springs and with changes in the vertical position of its members → replace the work of the springs and the work of the weights by negative of the respective potential energy changes

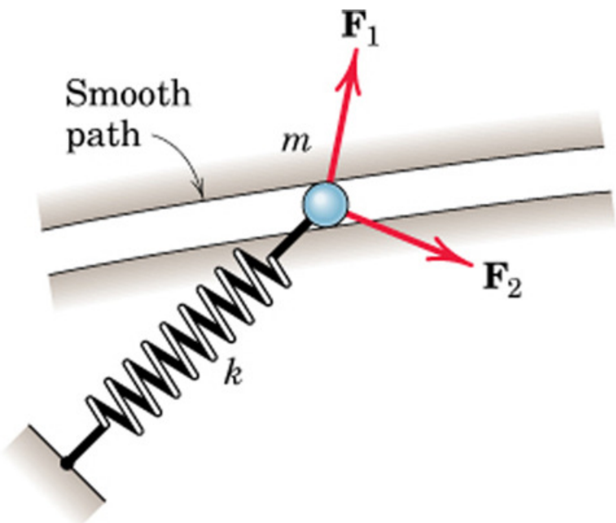
Total Virtual Work δU = work done by all active forces ($\delta U'$) other than spring forces and weight forces + the work done by the spring forces and weight forces, i.e., $-(\delta V_e + \delta V_g)$

$$\boxed{\delta U' - (\delta V_e + \delta V_g) = 0} \quad \text{or} \quad \boxed{\delta U' = \delta V}$$

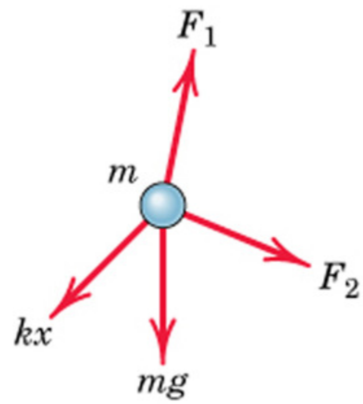
$V = V_e + V_g$ → Total Potential Energy of the system

Virtual Work

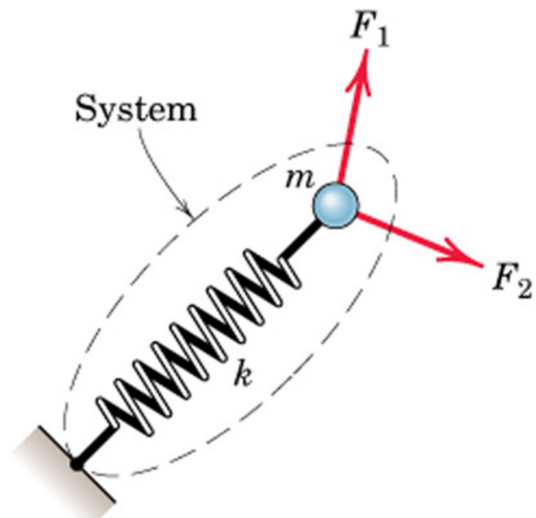
Active Force Diagrams: Use of two equations



$$\delta U = 0$$



$$\delta U' - (\delta V_e + \delta V_g) = 0$$



Principle of Virtual Work

The virtual work done by all external active forces (other than the gravitational and spring forces accounted for in the potential energy terms) on a mechanical system in equilibrium equals the corresponding change in the total elastic and gravitational potential energy of the system for any and all virtual displacements consistent with the constraints.

Virtual Work

Stability of Equilibrium

If work done by all active forces other than spring forces and weight forces is zero → $\delta U' = 0$

→ No work is done on the system by the non-potential forces

→ $\boxed{\delta(V_e + V_g) = 0}$ or $\boxed{\delta V = 0}$

→ Equilibrium configuration of a mechanical system is one for which the total potential energy *V* of the system has a stationary value.

→ For a SDOF system, it is equivalent to state that:

$$\boxed{\frac{dV}{dx} = 0}$$

→ A mechanical system is in equilibrium when the derivative of its total potential energy is zero

→ For systems with multiple DOF, partial derivative of *V* wrt each coordinate must be zero for equilibrium.

Virtual Work

Stability of Equilibrium: SDOF

Three Conditions under which this eqn applies when total potential energy is:

$$\frac{dV}{dx} = 0$$

→ Minimum (Stable Equilibrium)

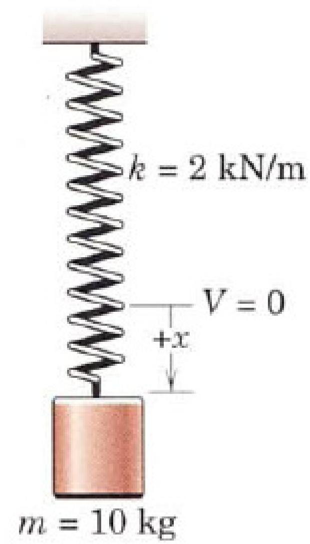
→ Maximum (Unstable Equilibrium)

→ Constant (Neutral Equilibrium)

Equilibrium	$\frac{dV}{dx} = 0$
Stable	$\frac{d^2V}{dx^2} > 0$
Unstable	$\frac{d^2V}{dx^2} < 0$

Example problem

The 10 kg cylinder is suspended by the spring, which has a stiffness of 2 kN/m. Show that potential energy is minimum at the equilibrium position



$$V_e = \frac{1}{2}kx^2 \qquad V_g = -mgx$$

$$V = V_e + V_g$$

$$\frac{dV}{dx} = 0 \qquad \frac{dV}{dx} = kx - mg = 0 \qquad x = mg/k$$

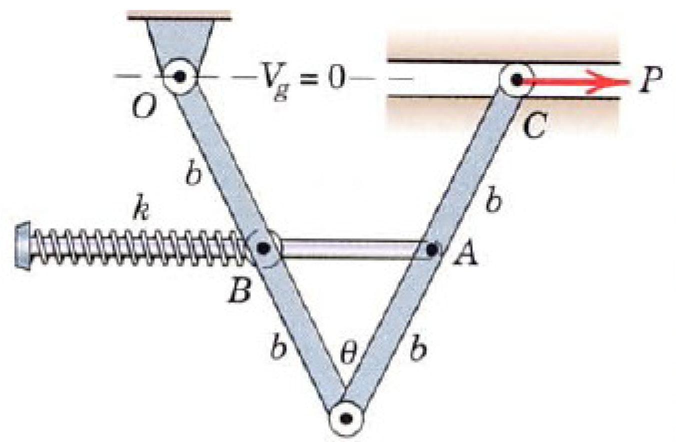
$$\frac{d^2V}{dx^2} = k \quad > 0$$ Minimum potential

The two uniform links, each of mass m, are in the vertical plane and are connected and constrained as shown. As the angle θ between the links increases with the application of the horizontal force P, the light rod, which is connected at A and passes through a pivoted collar at B, compresses the spring of stiffness k. If the spring is uncompressed in the position where $\theta = 0$, determine the force P which will produce equilibrium at the angle θ.

$[V_e = \frac{1}{2}kx^2]$ $\qquad V_e = \frac{1}{2}k\left(2b\sin\frac{\theta}{2}\right)^2 = 2kb^2\sin^2\frac{\theta}{2}$

$[V_g = mgh]$ $\qquad V_g = 2mg\left(-b\cos\frac{\theta}{2}\right)$

$$\delta U' = P\,\delta\left(4b\sin\frac{\theta}{2}\right) = 2Pb\cos\frac{\theta}{2}\,\delta\theta$$



$[\delta U' = \delta V_e + \delta V_g]$

$$2Pb\cos\frac{\theta}{2}\,\delta\theta = \delta\left(2kb^2\sin^2\frac{\theta}{2}\right) + \delta\left(-2mgb\cos\frac{\theta}{2}\right)$$

$$= 2kb^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,\delta\theta + mgb\sin\frac{\theta}{2}\,\delta\theta$$

$$\boxed{P = kb\sin\frac{\theta}{2} + \frac{1}{2}mg\tan\frac{\theta}{2}}$$